Effective Categoricity of Equivalence Structures

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Abstract

We investigate effective categoricity of computable equivalence structures $\mathcal{A}$. We show that $\mathcal{A}$ is computably categorical if and only if $\mathcal{A}$ has only finitely many finite equivalence classes, or $\mathcal{A}$ has only finitely many infinite classes, bounded character, and at most one finite k such that there are infinitely many classes of size k. We also prove that all computably categorical structures are relatively computably categorical, that is, have computably enumerable Scott families of existential formulas. Since all computable equivalence structures are relatively Δ_3^0 categorical, we further investigate when they are Δ_2^0 categorical. We also obtain results on the index sets of computable equivalence structures.

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1 Introduction

In computable model theory we are interested in effective versions of model theoretic notions and constructions. Here we study the computability theoretic bounds on the complexity of isomorphisms of structures within the same isomorphism type. We focus on equivalence structures. We consider only countable structures for computable languages, and for infinite structures we may assume that their universe is ω. We identify sentences with their Gödel codes. The *atomic diagram* of a structure $\mathcal{A}$ for L is the set of all quantifier-free sentences in L_A, L expanded by constants for the elements in A, which are true in $\mathcal{A}$. A structure is *computable* if its atomic diagram is computable. In other words, a structure $\mathcal{A}$ is computable if there is an algorithm that determines for every quantifier-free formula $\theta(x_0, \dots, x_{n-1})$ and every sequence $(a_0, \dots, a_{n-1}) \in A^n$, whether $\mathcal{A} \models \theta(a_0, \dots, a_{n-1})$. The *elementary diagram* of $\mathcal{A}$ is the set of all sentences of L_A that are true in $\mathcal{A}$. A structure $\mathcal{A}$ is *decidable* if its elementary diagram is computable. For $n > 0$, the *n-diagram* of $\mathcal{A}$ is the set of all Σ_n sentences of L_A that are true in $\mathcal{A}$. A structure is *n-decidable* if its n-diagram is computable.

A computable structure $\mathcal{A}$ is *computably categorical* if for every computable isomorphic copy $\mathcal{B}$ of $\mathcal{A}$, there is a *computable* isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. For example, the ordered set of rational numbers is computably categorical, while the ordered set of natural numbers is not. Moreover, Goncharov and Dzgoev [11], and Remmel [28] independently proved that a computable linear ordering is computably categorical if and only if it has only finitely many successors. Furthermore, Goncharov and Dzgoev [11], and Remmel [29] established that a computable Boolean algebra is computably categorical if and only if it has finitely many atoms (see also LaRoche [21]). Miller [26] proved that no computable tree of height ω is computably categorical. Lempp, McCoy, Miller, and Solomon [22] characterized computable trees of finite height that are computably categorical. Nurtazin [27], and Metakides and Nerode [24] established that a computable algebraically closed field of finite transcendence degree over its prime field is computably categorical. Goncharov [8] and Smith [31] characterized computably categorical abelian p-groups as those that can be written in one of the following forms: $(\mathbb{Z}(p^\infty))^l \oplus \mathcal{G}$ for $l \in \omega \cup \{\infty\}$ and $\mathcal{G}$ is finite, or $(\mathbb{Z}(p^\infty))^n \oplus \mathcal{G} \oplus (\mathbb{Z}(p^k))^\infty$, where $n, k \in \omega$ and $\mathcal{G}$ is finite. Goncharov, Lempp, and Solomon [14] proved that a computable, ordered, abelian group is computably categorical if and only if it has finite rank. Similarly, they showed that a computable, ordered, Archimedean group is computably categorical if and only if it has finite rank.

For any computable ordinal α, we say that a computable structure $\mathcal{A}$ is Δ^0_α *categorical* if for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is a Δ^0_α isomorphism form $\mathcal{A}$ onto $\mathcal{B}$. Lempp, McCoy, Miller, and Solomon [22] proved that for every $n \geq 1$, there is a computable tree of finite height, which is Δ^0_{n+1} categorical but not Δ^0_n categorical. We say that $\mathcal{A}$ is *relatively computably categorical* if for every structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an isomorphism that is computable relative to the atomic diagram of $\mathcal{B}$. Similarly, a computable $\mathcal{A}$ is *relatively* Δ^0_α *categorical* if for every $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an isomorphism that is Δ^0_α relative to the atomic diagram of $\mathcal{B}$. Clearly, a relatively Δ^0_α categorical structure is Δ^0_α categorical. We are especially interested in the case when $\alpha = 2$. McCoy [23] characterized, under certain restrictions, Δ^0_2 categorical and relatively Δ^0_2 categorical linear orderings and Boolean algebras. For example, a computable Boolean algebra is relatively Δ^0_2 categorical if and only if it can be expressed as a finite direct sum $c_1 \vee \ldots \vee c_n$, where each c_i is either atomless, an atom, or a 1-atom. Using an enumeration result of Selivanov [30], Goncharov [9] showed that there is a computable structure, which is computably categorical but not relatively computably categorical. Using a relativized version of Selivanov's enumeration result, Goncharov, Harizanov, Knight, McCoy, Miller, and Solomon [12] showed that for each computable successor ordinal α, there is a computable structure, which is Δ^0_α categorical but not relatively Δ^0_α categorical.

It is not known whether for a computable limit ordinal α, there is a computable structure that is Δ^0_α categorical but not relatively Δ^0_α categorical (see [12]). It is also not known whether for any computable successor ordinal α, there is a rigid computable structure that is Δ^0_α categorical but not relatively Δ^0_α categorical. Another open question is whether every Δ^1_1 categorical computable structure must be relatively Δ^1_1 categorical (see [13]).

There are syntactic conditions that are equivalent to relative Δ^0_α categoricity. These conditions involve the existence of certain families of formulas, that is, certain Scott families. Scott families come from Scott's Isomorphism Theorem, which says that for a countable structure $\mathcal{A}$, there is an $L_{\omega_1\omega}$ sentence whose countable models are exactly the isomorphic copies of $\mathcal{A}$. A *Scott family* for a structure $\mathcal{A}$ is a countable family Φ of $L_{\omega_1\omega}$ formulas, possibly with finitely many fixed parameters from A, such that:

(i) Each finite tuple in $\mathcal{A}$ satisfies some $\psi \in \Phi$;

(ii) If $\overrightarrow{a}$, $\overrightarrow{b}$ are tuples in $\mathcal{A}$, of the same length, satisfying the same formula in Φ, then there is an automorphism of $\mathcal{A}$ that maps $\overrightarrow{a}$ to $\overrightarrow{b}$.

A *formally c.e. Scott family* is a c.e. Scott family consisting of finitary existential formulas. A *formally* Σ^0_α *Scott family* is a Σ^0_α Scott family consisting of computable Σ_α formulas. Roughly speaking, computable infinitary formulas are $L_{\omega_1\omega}$ formulas in which the infinite disjunctions and conjunctions are taken over computably enumerable (c.e.) sets. We can classify computable formulas according to their complexity as follows. A computable Σ_0 or Π_0 formula is a finitary quantifier-free formula. Let $\alpha > 0$ be a computable ordinal. A computable Σ_α formula is a c.e. disjunction of formulas $(\exists \overrightarrow{u})\theta(\overrightarrow{x}, \overrightarrow{u})$, where θ is computable Π_β for some $\beta < \alpha$. A computable Π_α formula is a c.e. conjunction of formulas $(\forall \overrightarrow{u})\theta(\overrightarrow{x}, \overrightarrow{u})$, where θ is computable Σ_β for some $\beta < \alpha$. Precise definition of computable infinitary formulas involves assigning indices to the formulas, based on Kleene's system of ordinal notations (see [2]). The important property of these formulas is given in the following theorem due to Ash.

Theorem 1.1 *For a structure $\mathcal{A}$, if $\theta(\overrightarrow{x})$ is a computable Σ_α formula, then the set $\{\overrightarrow{a} : \mathcal{A} \models \theta(\overrightarrow{a})\}$ is Σ^0_α relative to the atomic diagram of $\mathcal{A}$.*

An analogous result holds for computable Π_α formulas.

It is easy to see that if $\mathcal{A}$ has a formally c.e. Scott family, then $\mathcal{A}$ is relatively computably categorical. In general, if $\mathcal{A}$ has a formally Σ^0_α Scott family, then $\mathcal{A}$ is relatively Δ^0_α categorical. Goncharov [9] showed that if $\mathcal{A}$ is 2-decidable and computably categorical, then it has a formally c.e. Scott family. Ash [1] showed that, under certain decidability conditions on $\mathcal{A}$, if $\mathcal{A}$ is Δ^0_α categorical, then it has a formally Σ^0_α Scott family. For the relative notions, the decidability conditions are not needed. Moreover, Ash, Knight, Manasse, and Slaman [3], and independently Chisholm [5] established the following result.

Theorem 1.2 *Let $\mathcal{A}$ be a computable structure. Then the following are equivalent:*

(a) A *is relatively Δ^0_α categorical;*

(b) A *has a formally Σ^0_α Scott family;*

(c) A *has a c.e. Scott family consisting of computable Σ_α formulas.*

Cholak, Goncharov, Khoussainov, and Shore [6] gave an example of a computable structure that is computably categorical, but ceases to be after naming any element of the structure. Such a structure is not relatively computably categorical. On the other hand, Millar [25] previously established that if a structure $\mathcal{A}$ is 1-decidable, then any expansion of $\mathcal{A}$ by finitely many constants remains computably categorical. Khoussainov and Shore [18] proved that there is a computably categorical structure without a formally c.e. Scott family whose expansion by any finite number of constants is computably categorical. A similar result was established by Kudinov by a different method. Using a modified family of enumerations constructed by Selivanov [30], Kudinov produced a computably categorical, 1-decidable structure without a formally c.e. Scott family.

A structure is *rigid* if it does not have nontrivial automorphisms. A computable structure is Δ^0_α *stable* if every isomorphism from $\mathcal{A}$ onto a computable structure is Δ^0_α. If a computable structure is rigid and Δ^0_α categorical, then it is Δ^0_α stable. A *defining family* for a structure $\mathcal{A}$ is a set Φ of formulas with one free variable and a fixed finite tuple of parameters from A such that:

(i) Every element of A satisfies some formula $\psi \in \Phi$;

(ii) No formula of Φ is satisfied by more than one element of A.

A defining family Φ is *formally* Σ^0_α if it is a Σ^0_α set of computable Σ_α formulas. In particular, a defining family Φ is *formally c.e.* if it is a c.e. set of finitary existential formulas. For a rigid computable structure $\mathcal{A}$, there is a formally Σ^0_α Scott family iff there is a formally Σ^0_α defining family.

In Section 2, we investigate algorithmic properties of computable equivalence structures, their equivalence classes, and their characters. In Section 3, we examine effective categoricity of equivalence structures. We characterize the computably categorical equivalence structures, and show that they are all relatively computably categorical. We show that $\mathcal{A}$ is computably categorical if and only if the following two conditions are satisfied:

(i) There is an upper bound on the size of the finite equivalence classes of $\mathcal{A}$;

(ii) There is at most one cardinal k such that $\mathcal{A}$ has infinitely many equivalence classes of size k.

In Section 4, we characterize relatively Δ_2^0 categorical equivalence structures as those with either finitely many infinite equivalence classes, or with an upper bound on the size of the finite equivalence classes. We also consider the complexity of isomorphisms for structures $\mathcal{A}$ and $\mathcal{B}$ such that both $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$ are computable, or Δ_2^0. Finally, we show that every computable equivalence structure is relatively Δ_3^0 categorical.

The notions and notation of computability theory are standard and as in Soare [32]. We fix $\langle \cdot, \cdot \rangle$ to be a computable bijection from ω^2 onto ω. Let $(W_e)_{e \in \omega}$ be an effective enumeration of all c.e. sets.

2 Computable equivalence structures

An equivalence structure $\mathcal{A} = (A, E^{\mathcal{A}})$ consists of a set with a binary relation that is reflexive, symmetric, and transitive. An equivalence structure $\mathcal{A}$ is *computable* if A is a computable subset of ω and E is a computable relation. If A is an infinite set (which is usual), we may assume, without loss of generality, that $A = \omega$. The $\mathcal{A}$-equivalence class of $a \in A$ is

$$[a]^{\mathcal{A}} = \{x \in A : xE^{\mathcal{A}}a\}.$$

We generally omit the superscript $^{\mathcal{A}}$ when it can be inferred from the context.

We will proceed from the simpler structures, which are computably categorical, to the more complicated ones, which are Δ_3^0 categorical but not Δ_2^0 categorical.

Definition 2.1 (i) *Let $\mathcal{A}$ be an equivalence relation. The* character of $\mathcal{A}$ *is the set*

$$\chi(\mathcal{A}) = \{\langle k, n \rangle : \ n, k > 0 \textit{ and } \mathcal{A} \textit{ has at least } n \textit{ equivalence classes of size } k\}.$$

(ii) *We say that $\mathcal{A}$ has* bounded character *if there is some finite k such that all finite equivalence classes of $\mathcal{A}$ have size at most k.*

Let

$$Inf^{\mathcal{A}} = \{a : [a]^{\mathcal{A}} \text{ is infinite}\} \text{ and } Fin^{\mathcal{A}} = \{a : [a]^{\mathcal{A}} \text{ is finite}\}.$$

The following lemmas will be needed.

Lemma 2.2 *For any computable equivalence structure $\mathcal{A}$:*

(a) $\{\langle k, a \rangle : card([a]^{\mathcal{A}}) \leq k\}$ *is a Π_1^0 set, and* $\{\langle k, a \rangle : card([a]^{\mathcal{A}}) \geq k\}$ *is a Σ_1^0 set;*

(*b*) $Inf^{\mathcal{A}}$ *is a* Π^0_2 *set, and* $Fin^{\mathcal{A}}$ *is a* Σ^0_2 *set;*

(*c*) $\chi(\mathcal{A})$ *is a* Σ^0_2 *set.*

Proof: (*a*) The condition $card([a]^{\mathcal{A}}) \leq k$ holds if and only if

$$(\forall x_1)\cdots(\forall x_{k+1})\,(\,(x_1 E a \;\&\; \cdots \;\&\; x_{k+1} E a) \Rightarrow \bigvee_{i,j \neq k+1} x_i = x_j\,)\,.$$

(*b*) We have $a \in Fin^{\mathcal{A}}$ if and only if

$$(\exists k)[card([a]^{\mathcal{A}}) = k],$$

and $a \in Inf^{\mathcal{A}}$ if and only if $a \notin Fin^{\mathcal{A}}$.

(*c*) We have $\langle k, n\rangle \in \chi(\mathcal{A})$ if and only if

$$(\exists x_1)\cdots(\exists x_n)\,(\,\bigwedge_i card([x_i]^{\mathcal{A}}) = k \;\&\; \bigwedge_{i \neq j} \neg(x_i E x_j)\,)\,.$$

□

We say that a subset K of $\omega \times \omega$ is a *character* if there is some equivalence structure with character K. This is the same as saying that $K \subseteq (\omega - \{0\}) \times (\omega - \{0\})$, for all $n > 0$ and k,

$$\langle k, n+1\rangle \in K \Rightarrow \langle k, n\rangle \in K.$$

Lemma 2.3 *For any* Σ^0_2 *character* K*, there is a computable equivalence structure* $\mathcal{A}$ *with character* K*, which has infinitely many infinite equivalence classes. Furthermore,* $Fin^{\mathcal{A}}$ *is a* Π^0_1 *set.*

Proof: Let R be a computable relation such that

$$\langle k, n\rangle \in K \iff (\exists w)(\forall y)R(k, n, w, y).$$

We will define a set B of quadruples $\langle k, n, w, z\rangle$ such that when we look at numbers only below z, we believe that w is the least witness that $\langle k, n\rangle \in K$, and such that for every other initial segments below z, there is some $v < w$ that could be such a witness. Define the set B as follows:

$$\begin{aligned} B &= \{\langle k, n, w, z\rangle : z > 0 \;\&\; (\forall y < z)R(k, n, w, y) \;\& \\ (\forall v &< w)(\exists y < z)\neg R(k, n, v, y) \;\& \\ (\forall z' &< z)(\exists v < w)(\forall y < z')R(k, n, v, y)\}. \end{aligned}$$

The set B is a computable subset of ω with an infinite complement.

The equivalence structure $\mathcal{A}$ will consist of one class for each element of B, together with an infinite family of infinite equivalence classes. Partition

$\omega \setminus B$ into two disjoint, infinite, computable subsets. Use the first subset to define the infinitely many infinite classes, and let the second subset be $C = \{c_0, c_1, \dots\}$. The classes with representatives from B are defined in stages. Let $B = \{b_0, b_1, \dots\}$, and let $b_i = \langle k_i, n_i, w_i, z_i \rangle$.

At *stage* 0, we put $\{c_0, \dots, c_{k_0-2}\}$ into the equivalence class of b_0.

After stage s, we have s equivalence classes with representatives $b_0, \dots, b_{s-1}$. For $i < s$, some classes with representatives b_i have size k_i, and others have been declared to be infinite and have at least s elements. These partial classes contain elements $c_0, \dots, c_{p(s)}$ from C.

At *stage* $s+1$, we check for all $i \leq s$ and all $y \leq s$ whether $R(k_i, n_i, w_i, y)$. If the class $[b_i]$ has previously been declared to be infinite, we simply add one new element to this class. For any other i such that some $R(k_i, n_i, w_i, y)$ does not hold with $y \leq s$, we declare $[b_i]$ to be infinite and add s new elements from C to $[b_i]$. If $[b_{s+1}]$ is not declared to be infinite, then we put $k_{s+1} - 1$ new elements from C into this class.

Let $\mathcal{A}$ be the structure constructed by this process. It is clear that whenever $\langle k, n \rangle \in K$, then there will be a unique $b_i = \langle k, n, w_i, z_i \rangle$ such that $[b_i]$ has size k, and that these are the only finite classes of $\mathcal{A}$. Thus, $\chi(\mathcal{A}) = K$. If $\langle k, n \rangle \notin K$, then the class $[b_i]$ corresponding to $\langle k, n \rangle$ will be infinite. Hence $\mathcal{A}$ has infinitely many infinite equivalence classes.

We note that for any pair (k, n), there is at most one w such that for all z, we have $(k, n, w, z) \in R$, and such that for some $\tilde{z}$, we have $\langle k, n, w, \tilde{z} \rangle \in B$. Also, for each triple (k, n, w), there is at most one z such that $\langle k, n, w, z \rangle \in B$, and for each triple (k, n, z), there is at most one w such that $\langle k, n, w, z \rangle \in B$. Now it is clear that $[b_i]$ is finite if and only if $(\forall z) R(k_i, n_i, w_i, z)$, and $[b_i]$ has k_i elements. Furthermore, if $\langle k, n \rangle \in K$, then

$$card(\{[b_i] : card[b_i] = k\}) \geq n.$$

Then for $c \notin B$, $[c]$ is finite if and only if

$$(\forall i)(cEb_i \Rightarrow [b_i] \text{ is finite}).$$

This completes the proof. □

Lemma 2.4 *For any $r \leq \omega$ and any bounded character K (whether $K \in \Sigma^0_2$ or not), there is a computable equivalence structure $\mathcal{A}$ with character K, which has exactly r infinite equivalence classes. Furthermore, $Fin^{\mathcal{A}}$ is a computable set.*

Proof: The desired structure $\mathcal{A}$ will have three components, each itself either finite or computable. First, there will be r infinite equivalence classes. Second, there will be a finite set $\{k_1, \dots, k_m\} \subset \omega$, and an infinite family of equivalence classes of size k_i for $i = 1, \dots, m$. Third, there will be a finite set $\{j_1, \dots, j_p\} \subset \omega$ with corresponding natural numbers $n_i > 0$ for $i = 1, \dots, p$ and j_i equivalence classes of size n_i. It is clear that a computable structure $\mathcal{A}$ can be constructed such that each desired component is computable. □

The proof of Lemma 2.3 really needed the assumption of infinitely many infinite equivalence classes, since it is possible that either finitely many or infinitely many infinite equivalence classes come from B.

If there are just finitely many infinite equivalence classes, then the notions of an s-function and an s_1-function are important. These functions were introduced by Khisamiev in [15]. The s-functions are called *limitwise monotonic* in [17].

Definition 2.5 *Let $f : \omega^2 \to \omega$. The function f is an s-*function *if the following hold:*

1. *For every i and s, $f(i,s) \leq f(i,s+1)$;*
2. *For every i, the limit $m_i = lim_s f(i,s)$ exists.*

   *We say that f is an s_1-*function *if, in addition:*

3. *For every i, $m_i < m_{i+1}$.*

Lemma 2.6 *Let $\mathcal{A}$ be a computable equivalence structure with finitely many infinite equivalence classes and infinite character.*

(a) There exists a computable s-function f with corresponding limits $m_i = lim_s f(i,s)$ such that $\langle k,n\rangle \in \chi(\mathcal{A})$ if and only if

$$card(\{i : k = m_i\}) \geq n.$$

(b) If the character is unbounded, then there is a computable s_1-function f such that $\mathcal{A}$ contains an equivalence class of size m_i for all i, where

$$m_i = lim_s f(i,s).$$

Proof: We may assume, without loss of generality, that $\mathcal{A}$ has no infinite equivalence classes, since the infinite classes can be captured by a finite set of representatives.

(a) Define a computable sequence of representatives for all equivalence classes of $\mathcal{A}$ by setting $a_0 = 0$, and setting a_{i+1} to be the least $a > a_i$ such that $\neg(aEa_j)$ for all $j \leq i$. Now simply let

$$f(i,s) = card(\{a \leq s : aEa_i\}).$$

(b) We will define a uniformly computable family $(a_i^s)_{i,s}$ for $i \leq s$ in such a way that $a_i = lim_s a_i^s$ exists. We will also define a computable sequence $(p_s)_s$ and let

$$f(i,s) = card(\{a \leq p_s : aEa_i^s\}).$$

Hence we will have

$$m_i = lim_s(card(\{a \leq s : aEa_i\}) = card([a_i])).$$

At *stage* 0, we have $p_0 = 0$ and $a_0^0 = 0$, so $f(0,0) = 1$.

After stage s, we have p_s and $a_0^s, \dots, a_s^s$ such that

$$f(i,s) = card(\{a \leq p_s : aEa_i^s\}),$$

and

$$f(0,s) < f(1,s) < \cdots < f(s,s).$$

At *stage* $s+1$, we look for the least $p > p_s$ and the lexicographically least sequence $b_0, \dots, b_{s+1} < p$ such that for all $i \leq s$,

$$f(i,s) \leq card(\{a \leq p : aEb_i\}) < card(\{a \leq p : aEb_{i+1}\}),$$

and, furthermore, $b_i = a_i^{s+1}$ whenever there do not exist a, j with $j < i$, aEa_j^s, and $p_s < a \leq p$. Then we let $a_i^{s+1} = b_i$ for each i, and let $p_{s+1} = p$.

To see that such p exists, simply let m be the largest number such that $[a_j^s] = \{a \leq p : aEa_j^s\}$ for all $j \leq m$, and let $b_i = a_i^s$ for all $i \leq m$. Then use the fact that $\chi(\mathcal{A})$ is unbounded to find $b_{m+1}, \dots, b_{s+1}$ with

$$\begin{aligned} card([a_m^s]) &< card([b_{m+1}]) < card([b_{m+2}]) < \cdots \\ &< card([b_{s+1}]), \end{aligned}$$

and take p large enough so that $[b_i] = \{a \leq p : aEb_i\}$. □

Lemma 2.7 *For any computable s_1-function f, $\{lim_s f(i,s) : i \in \omega\}$ is a Δ_2^0 set.*

Proof: Let $m_i = lim_s f(i,s)$. Since $m_0 < m_1 < \dots$, it follows that $m \in \{m_i : i \in \omega\}$ if and only if there exists $i \leq m$ such that $m = m_i$, which has the following two characterizations:

$$(\exists s)(\forall t > s) f(i,t) = m,$$

and

$$(\forall s)(\exists t > s) f(i,t) = m.$$

Thus, $\{m_i : i \in \omega\}$ is both Σ_2^0 and Π_2^0. □

Lemma 2.8 *Let K be a Σ_2^0 character, and let r be finite.*

(*a*) *Let f be a computable s-function with the corresponding limits $m_i = lim_s f(i,s)$ such that*

$$\langle n,k \rangle \in K \iff card(\{i : k = m_i\}) \geq n.$$

Then there is a computable equivalence structure $\mathcal{A}$ with $\chi(\mathcal{A}) = K$ and with exactly r infinite equivalence classes.

(*b*) *Let f be a computable s_1-function with corresponding limits $m_i = lim_s f(i,s)$ such that $\langle m_i, 1 \rangle \in K$ for all i. Then there is a computable equivalence structure $\mathcal{A}$ with $\chi(\mathcal{A}) = K$ and exactly r infinite equivalence classes.*

Proof: Clearly, it suffices to prove the statements for $r = 0$. We may assume that $f(i, 0) \geq 1$ for all i.

(a) Let $a_i = 2i$. We will build an equivalence structure $\mathcal{A}$ with equivalence classes $[a_i]$ of size m_i. At stage 0, make the elements $1, 3, \ldots, (2f(0, 0) - 1)$ equivalent to a_0. After stage s, we have exactly $f(m, s)$ elements equivalent to a_m for each $m \leq s$. Then we add $f(m, s+1) - f(m, s)$ elements to $[a_m]$ for $m \leq s$, and put $f(s+1, s+1) - 1$ elements into the class of a_{s+1}.

(b) Since there is an s_1-function, the character K must be unbounded. We modify the argument for Lemma 2.3 as follows. The pool of elements to put into the equivalence classes is now simply $\omega \setminus B$.

Here is the first modification. When we find $\neg R(k_i, n_i, w_i, z)$ for some z at stage $s + 1$, we can no longer create an infinite equivalence class, but we have already put k_i elements in the equivalence class of b_i. So we will set this bloc $[b_i]$ aside until we find a number j and a stage s such that $k_i \leq f(j, s)$. Since there is an increasing sequence $m_0 < m_1 < \cdots$ corresponding to the s_1-function, such j and s will eventually be found. Then we will assign a marker j to b_i, and add $f(j, s) - k_i$ elements to the bloc to create an equivalence class with $f(j, s)$ elements. If at a later stage t we have $f(j, t) > f(j, s)$, then we will add $f(j, t) - f(j, s)$ more elements to the class. Since $lim_s f(j, s) = m_j$, we will eventually have an equivalence class of size m_j.

This means that we may have created an extra equivalence class with $f(j, s)$ elements, so the second modification is that when we create a class with $k = f(j, s)$ elements, we may need (perhaps temporarily) to remove from our construction any class corresponding to $\langle k, 1, w, z\rangle$. That is, we set these (finitely many) blocs aside to be put into a larger class, just as if we had found that $\neg R(k, 1, w, z)$, but we make a note that they may need to be revived later. If at some later stage t, we find k' such that

$$k' = f(j, t) > f(j, s) = k,$$

so that we will increase the size of the class with marker j, then we are going to remove the classes corresponding to $\langle k', 1\rangle$ and at the same time revive the classes corresponding to $\langle k, 1\rangle$. At this stage, we remove the attachment to $f(j, s)$ of the bloc, and check for all $b_i = \langle k, 1, w, z\rangle$:

(1) Whether $R(k, 1, w, z)$ still holds for all $z \leq t$;

(2) Whether the bloc corresponding to b_i has been put into a larger class yet.

If (1) is false and (2) is true, then there is nothing else to do. If (1) and (2) are both false, then we keep the bloc aside for later use. If (1) is true and (2) is false, then we revive this bloc. If both (1) and (2) are true, then we create a new class with k elements for each $\langle k, 1, w, z\rangle$ such that $R(k, 1, w, z)$ holds for all $z \leq t$.

We will now describe the construction in detail. Set $C_{-1} = \omega - B$. No j-markers are used at stage -1. If $b_i = \langle k_i, n_i, w_i, z_i\rangle$ as in Lemma 2.3, let us say that b_i is *active* at stage s if for all $z \leq s$, $R(k_i, n_i, w_i, z)$, and otherwise we say that b_i is *inactive*.

After s stages, we will have some equivalence classes with active representatives $b_i = \langle k, n, w, z \rangle$ or revived representatives b_i', containing k elements, such that for all $z \leq s$, $R(k, n, w, z)$. We will have some equivalence classes containing $f(j, s)$ elements corresponding to the s_1-function f. There will also be certain blocs of size k_i corresponding to inactive b_i, and certain displaced blocs corresponding to active b_i, which are waiting to be put into a larger equivalence class. Finally, there are some active $b_i = \langle k_i, 1, w, z \rangle$, which have been displaced by some equivalence class of size $f(j, s) = k_i$, but may need to be revived. There is a pool C_s of remaining elements that may be used to fill out new equivalence classes. At stage $s + 1$, we perform the following steps.

First, we check whether $b_{s+1} = \langle k, n, w, z \rangle$ is active at stage $s + 1$. If so, then we check whether $n = 1$ and $k = f(j, s + 1)$ for some current equivalence class with marker j. If such j exists, then we put b_{s+1} into the pool C_{s+1}. Otherwise, we create an equivalence class with k elements consisting of b_{s+1} and $k - 1$ elements from the pool C_s. If b_{s+1} is already inactive, then we simply add it to the pool C_{s+1}.

Second, we check for $i \leq s$ whether some b_i that was active at stage s becomes inactive at stage $s + 1$. If such b_i was representing an equivalence class at stage s, then that class is set aside as a bloc to be attached to some $f(j, t)$ at a later stage.

Third, we look for the smallest bloc that has been set aside for some inactive b_i at stage s, and check whether there exists a previously unused $j \leq s + 1$ such that $k_i \leq f(j, s + 1)$. If so, then we create an equivalence class including this bloc and containing $f(j, s + 1)$ elements.

Fourth, for all markers j that are being used at stage s, we check whether $f(j, s+1) > f(j, s)$. If so, then we add $f(j, s+1) - f(j, s)$ elements to the corresponding equivalence class. We then displace any class with a representative b_i for $i \leq s$ such that $k_i = f(j, s+1)$ and $n_i = 1$. That is, we set aside this class as a bloc to be attached later to some $f(j', t)$. Finally, we revive any active b_i such that $k_i = f(j, s)$ and $n_i = 1$, which was displaced by $f(j, s)$. This means that we create a completely new class with k_i elements and a new representative b_i' taken from the pool.

It is clear that eventually all elements from the pool are put into some equivalence class. It needs to be verified that this class eventually stabilizes at some finite size k, and that the resulting equivalence structure has the desired character $\chi(\mathcal{A})$.

Suppose that a is first put into some class attached to marker j at stage s. Then this class will have size $f(j, t)$ at any later stage t and will stabilize with m_j elements. Next, suppose that a is first put into some bloc with representative b_i or b_i'. There are two cases. If b_i remains active at all stages and is never displaced by any $f(j, t)$, then this class has exactly k_i elements at all future stages. Otherwise, this class is set aside as a bloc at some later stage, and then eventually put into a class with some marker j, which will stabilize with m_j elements. This is guaranteed by the fact that there are infinitely many $m_j > k_i$, and eventually the bloc containing b_i will have the highest priority. Thus, all equivalence classes stabilize at some finite size. Hence $\mathcal{A}$ has no infinite equivalence classes.

Now, fix a finite k and suppose that $\langle k, n\rangle \in K$ for all $n < r$, where $r \le \infty$. We need to verify that there are exactly r classes in $\mathcal{A}$ of size k. For each $n < r$, there will be a unique representative $b_i = \langle n, k, w, z\rangle$ that remains active at all stages, where w is the least element such that $(\forall z)R(n, k, w, z)$. For $n > 1$, this b_i will represent a class of size k, and can never be displaced. For $n = 1$, there are two possibilities. There can be some (unique) marker j such that $m_j = k$, which corresponds to a class stabilizing at size k. In this case, any classes corresponding to b_i (or any later b'_i) will be displaced and eventually not revived (once $f(j, s)$ converges to m_j). On the other hand, if there is no such marker j, then eventually there will be a unique class with representative b_i (or some b'_i) with k elements, which is never displaced. Classes represented by other b_p or by other markers can never have size k. Thus, $\chi(\mathcal{A}) = K$. □

The necessity of the computable s_1-function follows from the next result.

Theorem 2.9 *There is an infinite Δ^0_2 set D such that for any computable equivalence structure $\mathcal{A}$ with unbounded character K and no infinite equivalence classes, $\{k : \langle k, 1\rangle \in K\}$ is not a subset of D. Hence, for any computable s_1-function f with $m_0 < m_1 < \cdots$, where $m_n = lim_s f(n, s)$, there exists i such that $m_i \notin D$.*

Proof: We use a method similar to Post's construction of a simple set. Let $\mathcal{A}_e$ be the structure with universe ω and relation E_e defined by

$$iE_ej \iff \langle i, j\rangle \in W_e.$$

Let $[a]_e = \{j : aE_ej\}$. Then every computable equivalence structure is $\mathcal{A}_e$ for some e, and $[a]_e$ is the equivalence class of a. Define a c.e. relation R by

$$R(e, a) \iff card([a]_e) > 2e.$$

Then, by the standard uniformization theorem for c.e. relations (see Soare [32], p. 29), there exists a partial computable function f, called a selector for R, such that for every e,

$$(\exists a)R(e, a) \Longrightarrow R(e, f(e)).$$

Define D as follows:

$$k \in D \iff (\forall e < \frac{k}{2})(card([f(e)]_e) \neq k).$$

Then D is a Δ^0_2 set by part (a) of Lemma 2.2. For any ℓ, the set

$$\hat{D} = \{n : (\exists x < \ell)(n = card([f(x)]_x))\}$$

has cardinality at most ℓ, so that at most ℓ of the elements from the set $\{0, 1, \ldots, 2\ell\}$ may be in $\hat{D}$. Thus, the complement of D contains at most e elements from $\{0, 1, \ldots, 2e\}$. Hence D is infinite.

Now, suppose that $\mathcal{A}$ has unbounded character and has no infinite equivalence classes. Choose e so that $\mathcal{A} = \mathcal{A}_e$. Since $\chi(\mathcal{A})$ is unbounded, there exists a such that $R(e, a)$, so $a = f(e)$. Since $\mathcal{A}$ has no infinite equivalence classes,

$$card([a]^{\mathcal{A}}) = card([f(e)]_e) = k > 2e.$$

Then, by definition, $\langle k, 1\rangle \in \chi(\mathcal{A})$, but $k \notin D$.

Now, let f be any computable s_1-function. Let $m_i = lim_s f(i, s)$, and $K = \{\langle m_i, 1\rangle : i \in \omega\}$. Then there is an equivalence structure $\mathcal{A}$ with character K. Therefore, $m_i \notin D$ for some i. □

We note that in [17], Khoussainov, Nies, and Shore give a direct construction of a Δ^0_2 set that is not the range of an s-function. This fact allows an alternative proof of Theorem 2.9.

3 Computable categoricity of equivalence structures

We first investigate relative computable categoricity of computable equivalence structures by showing that they have a formally c.e. Scott family.

Proposition 3.1 *If $\mathcal{A}$ is a computable equivalence structure with only finitely many finite equivalence classes, then $\mathcal{A}$ is relatively computably categorical.*

Proof: Choose parameters $c_1, \ldots, c_n$, which are representatives of the n finite equivalence classes. A Scott formula for any finite sequence $\overrightarrow{a} = a_1, \ldots, a_m$ of elements from A is a conjunction of three formulas. The first formula is simply the conjunction of the formulas $x_i = x_j$ where $a_i = a_j$, and the formulas $\neg(x_i = x_j)$ where $a_i \neq a_j$. The second formula $\phi(\overrightarrow{x})$ is the conjunction of all formulas $x_i E x_j$ (when $a_i E^{\mathcal{A}} a_j$) and $\neg(x_i E x_j)$ (when it is not the case that $a_i E^{\mathcal{A}} a_j$). The third formula $\psi(\overrightarrow{x}, \overrightarrow{c})$ is the conjunction of all formulas $x_i E c_j$ (when $a_i E^{\mathcal{A}} c_j$) and $\neg(x_i E c_j)$ (when it is not the case that $a_i E^{\mathcal{A}} c_j$). It is clear that every tuple of elements from A satisfies one of these formulas.

Suppose that $\overrightarrow{a}$ and $\overrightarrow{b}$ satisfy the same Scott formula. Then, in particular, we have

$$a_i E^{\mathcal{A}} a_j \iff b_i E^{\mathcal{A}} b_j.$$

For any tuple $\overrightarrow{d}$, the equivalence class $[d_i]$ is finite if and only if some formula of the form $x_i E c_j$ occurs in the Scott formula of $\overrightarrow{d}$, and $[d_i]$ is infinite otherwise. We will define an automorphism H of $\mathcal{A}$ mapping $\overrightarrow{a}$ to $\overrightarrow{b}$.

For any equivalence class $[a]$ containing none of the elements of $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$, the function H will simply be the identity map. We also define $H(a_i) = b_i$ and $H(c_i) = c_i$. This induces a partial 1-1 function from the equivalence classes of $\mathcal{A}$ into the equivalence classes of $\mathcal{A}$, which fixes finite classes setwise and takes infinite classes to infinite classes. Within a particular finite class $[a_i]$ of size n, the partial function from $[a_i]$ to $[b_i]$ defined on $[a_i] \cap \{\overrightarrow{a}\}$ can be extended to an isomorphism from $[a_i]$ onto $[b_i]$.

For the infinite classes (whether finitely or infinitely many) the partial isomorphism of the classes may similarly be extended to a total isomorphism of the classes. Likewise, the partial function taking a_i to b_i may be extended to map the infinite class $[a_i]$ to the infinite class $[b_i]$. □

Proposition 3.2 *Let $\mathcal{A}$ be a computable equivalence structure with finitely many infinite classes, with bounded character, and with at most one finite k such that there are infinitely many equivalence classes of size k. Then $\mathcal{A}$ is relatively computably categorical.*

Proof: Let $c_1, \dots, c_n$ be representatives for the finite classes not of size k, and let $d_1, \dots, d_p$ be representatives for the finitely many infinite classes. Then the Scott formula for a finite sequence $\overrightarrow{a}$ from $\mathcal{A}$ is the conjunction of three formulas, the first two as in the proof of Proposition 3.1, and the third is the conjunction of all formulas $x_i E d_j$ (when $a_i E^{\mathcal{A}} d_j$) and $\neg(x_i E d_j)$ (when it is not the case that $a_i E^{\mathcal{A}} d_j$). Then $[a_i]$ is infinite if and only if $a_i E^{\mathcal{A}} d_j$ for some j, and $card([a_i]) = k$ if and only if $\neg(a_i E^{\mathcal{A}} d_j)$ for all $j \in \{1, \dots, p\}$, and also $\neg(a_i E^{\mathcal{A}} c_j)$ for all $j \in \{1, \dots, n\}$.

Suppose that $\overrightarrow{a}$ and $\overrightarrow{b}$ satisfy the same Scott formula. Then we can define an automorphism of $\mathcal{A}$ extending the partial function which takes a_i to b_i as in the proof of Proposition 3.1. □

Corollary 3.3 *Let $\mathcal{A}$ be a computable equivalence structure of one of the following types:*

1. *$\mathcal{A}$ has only finitely many finite equivalence classes;*
2. *$\mathcal{A}$ has finitely many infinite classes, bounded character, and at most one finite k such that there are infinitely many classes of size k.*

Then $\mathcal{A}$ is relatively computably categorical.

In the remainder of this section, we will show that no other equivalence structures are computably categorical. For structures $\mathcal{A}$ with $Fin^{\mathcal{A}}$ computable there is a stronger result.

Proposition 3.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic computable equivalence structures such that $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$ are computable, and such that $\mathcal{A}$ has infinitely many equivalence classes of size k for at most one finite k. Then $\mathcal{A}$ and $\mathcal{B}$ are computably isomorphic.*

Proof: By Proposition 3.1, $Inf^{\mathcal{A}}$ and $Inf^{\mathcal{B}}$ are computably isomorphic, and, by Proposition 3.2, $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$ are computably isomorphic. □

Here is the first case of non-computable categoricity.

Theorem 3.5 *Suppose that there exist $k_1 < k_2 \leq \omega$ such that the computable equivalence structure $\mathcal{A}$ has infinitely many equivalence classes of size k_1 and infinitely many classes of size k_2. Then $\mathcal{A}$ is not computably categorical.*

Proof: We will define structures $\mathcal{C}$ and $\mathcal{D}$, both isomorphic to $\mathcal{A}$, such that $\{a : card([a]^{\mathcal{C}}) = k_1\}$ is a computable set, but

$$\{a : card([a]^{\mathcal{D}}) = k_1\}$$

is not computable. Then these two structures are not computably isomorphic, so $\mathcal{A}$ is not computably categorical.

Observe that $\chi(\mathcal{A})$ is a Σ^0_2 set by part (c) of Lemma 2.2, and, therefore, the set

$$K = \chi(\mathcal{A}) \setminus \{\langle k_1, n\rangle : n < \omega\}$$

is also Σ^0_2. Thus, if $\chi(\mathcal{A})$ is bounded, then there is a computable equivalence structure with character K and the same number of infinite equivalence classes as $\mathcal{A}$. If $\chi(\mathcal{A})$ is unbounded, then there are two cases.

First suppose that $\mathcal{A}$ has finitely many infinite equivalence classes. Then by Lemma 2.6, there is a computable s_1-function f for $\chi(\mathcal{A})$. Let $m_i = lim_s f(i,s)$. If $k_1 \neq m_i$ for any i, then f will be an s_1-function for the character K. If $k_1 = m_i$, we define a new s_1-function g for K by setting $g(j) = f(j)$ for $i < j$, and $g(j) = f(j+1)$ for $i \geq j$. Then there is a computable structure with character K by Lemma 2.8.

Next suppose that $\mathcal{A}$ has infinitely many infinite equivalence classes. Then, since K is Σ^0_2, Lemma 2.3 implies that there is a computable structure with character K and with infinitely many infinite equivalence classes.

In either case, we can now define a structure $\mathcal{C} \simeq \mathcal{A}$ (that is, with character $\chi(\mathcal{A})$) by setting

$$(2a+1)\ E^{\mathcal{C}}\ (2b+1) \iff aE^{\mathcal{B}}b,$$

and

$$(2(mk_1+i))\ E^{\mathcal{C}}\ (2(nk_1+j)) \iff m = n,$$

where $i, j < k_1$. In this structure, $\{a : card([a]^{\mathcal{C}}) = k_1\}$ is a computable set.

At the same time, we can build a structure $\mathcal{D} \simeq \mathcal{A}$ such that $\{a : card([a]^{\mathcal{D}}) = k_1\}$ is not computable. There are two cases, depending on whether k_2 is finite.

First suppose that k_2 is finite. We will build a computable structure $\mathcal{C}'$ with character

$$\{\langle k_1, n\rangle : n < \omega\} \cup \{\langle k_2, n\rangle : n < \omega\}$$

in which $\{a : card([a]) = k_1\}$ is a complete c.e. set, as follows. Let M be a complete c.e. set. Note that M is both infinite and co-infinite. The equivalence classes of $\mathcal{C}'$ will have representatives $2i$ so that $card([2i]) = k_1$ if $i \notin M$, and $card([2i]) = k_2$ if $i \in M$. The odd numbers will act as a pool of elements to fill out the classes.

The construction of $\mathcal{C}'$ is in stages. After stage s, there will be classes C^s_i containing $2i$, which will have k_1 elements if $i \notin M_s$, and k_2 elements if $i \in M_s$. At stage $s+1$, we add a new class containing $2s+2$, and also containing k_1-1 new odd elements from the pool if $s+1 \notin M_{s+1}$, and containing k_2-1 new elements from the pool if $s+1 \in M_{s+1}$. Also, for any $i \leq s$ such that $i \in M_{s+1} \setminus M_s$, we will add $k_2 - k_1$ new elements from the pool to the class $[2i]$.

Next suppose that $k_2 = \omega$. Just modify the construction above so that when $i \in M_s$, the class $[2i]$ contains $max\{k_1, s\}$ elements. The details are left to the reader.

Finally, we combine $\mathcal{A}$ and $\mathcal{C}'$ into $\mathcal{D}$ by coding $\mathcal{A}$ using odd numbers and $\mathcal{C}'$ using even numbers. Then

$$card([2a]^{\mathcal{D}}) = k_1 \iff a \notin M,$$

so that $\{d : card([d]^{\mathcal{D}}) = k_1\}$ is not computable. □

We observe that in the proof of Theorem 3.5, if $Fin^{\mathcal{A}}$ is computable, then in the case when $k_2 < \omega$, $Fin^{\mathcal{C}}$ and $Fin^{\mathcal{D}}$ will also be computable. Thus we have the following proposition.

Proposition 3.6 *For any $k_1 < k_2 < \omega$, and any computable equivalence structure $\mathcal{A}$ with $Fin^{\mathcal{A}}$ computable, with infinitely many equivalence classes of size k_1, and infinitely many equivalence classes of size k_2, there is a computable equivalence structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ with $Fin^{\mathcal{B}}$ computable such that $\mathcal{A}$ and $\mathcal{B}$ are not computably isomorphic.*

For the next result, we want to consider the so-called isomorphism problem for a class of structures. For a total computable function $\phi_e : \omega \times \omega \to \{0, 1\}$, let $\mathcal{C}_e = (\omega, \equiv_e)$ be the structure with

$$m \equiv_e n \iff \phi_e(\langle m, n\rangle) = 1.$$

It is easy to check that $\{e : \mathcal{C}_e \text{ is an equivalence structure}\}$ is a Π^0_2 set.

Convention: We say that a set is D^0_3 if it is a difference of two Σ^0_3 sets.

The following lemma is immediate from Lemma 2.2.

Lemma 3.7 (*a*) *For any finite r,*
$\{e : \mathcal{C}_e$ has at least r infinite equivalence classes$\}$ is a Σ^0_3 set.

(*b*) *For any finite r, $\{e : \mathcal{C}_e$ has exactly r infinite equivalence classes$\}$ is a D^0_3 set.*

(*c*) *The set $\{e : \mathcal{C}_e$ has infinitely many infinite equivalence classes$\}$ is Π^0_4.*

We need to look at the indices for Σ^0_2 sets in general. Let $(S_e)_{e\in\omega}$ be an enumeration of the Σ^0_2 sets, that is,

$$n \in S_e \iff (\exists m)(\langle m, n\rangle \notin W_e).$$

Then an enumeration $(K_e)_{e\in\omega}$ of the Σ^0_2 characters may be defined by

$$\langle k, n\rangle \in K_e \iff (\forall j \leq n)(\langle k, j\rangle \in S_e).$$

Lemma 3.8 *Let K be a fixed infinite Σ_2^0 character. Then the index set $\{e : K_e = K\}$ is Π_3^0 complete.*

Proof: The set $\{e : K_e = K\}$ is clearly Π_3^0. Now, let P be an arbitrary Π_3^0 set. Let S be a Σ_2^0 set such that

$$e \in P \iff (\forall k)(\langle k, e\rangle \in S),$$

where we may assume, without loss of generality, that

$$\langle k+1, e\rangle \in S \Rightarrow \langle k, e\rangle \in S.$$

Next, we will define a computable function f such that

$$K_{f(e)} = K \iff e \in P.$$

Set

$$\langle k, n\rangle \in K_{f(e)} \iff \langle k, n\rangle \in K \ \&\ \langle k, e\rangle \in S \ \&\ \langle n, e\rangle \in S.$$

If $e \in P$, then $\langle k, e\rangle \in S$ for all k, so that for every k and n,

$$\langle k, n\rangle \in K_{f(e)} \iff \langle k, n\rangle \in K.$$

If $e \notin P$, then there is some k_0 such that for all $k \geq k_0$, $\neg S(k, e)$. Since K is infinite, there is some $\langle k, n\rangle \in K$ such that either $k \geq k_0$ or $n \geq k_0$, and, therefore, $\langle k, n\rangle \notin K_{f(e)}$. □

We note that in the proof of Lemma 3.8, $K_{f(e)} \subseteq K$ for all e.

For a finite character K, the index set $\{e : K_e = K\}$ is Π_2^0 complete.

Theorem 3.9 *Let $\mathcal{A}$ be a computable equivalence structure with character K such that there does not exist a computable equivalence structure $\mathcal{B}$ with character K and with finitely many infinite equivalence classes. Then the index set $\{e : \mathcal{C}_e \simeq \mathcal{A}\}$ is Π_3^0 complete.*

Proof: The set $\{e : \mathcal{C}_e \simeq \mathcal{A}\}$ is Π_3^0 by Lemmas 2.2 and 3.8, since $\mathcal{C}_e \simeq \mathcal{A}$ if and only if $\chi(\mathcal{C}_e) = K$.

For the completeness, let the computable function f be as in the proof of Lemma 3.8. Use the technique of Lemma 2.3 uniformly to create the equivalence structure $\mathcal{C}_{g(e)}$ with character $K_{f(e)}$ and infinitely many infinite equivalence classes. Then $\mathcal{C}_{g(e)}$ is isomorphic to $\mathcal{A}$ if and only if $K_{f(e)} = K$. The result now follows from Lemma 3.8. □

Theorem 3.10 *Let K be an unbounded Σ_2^0 character. Let $\mathcal{A}$ be a computable equivalence structure with character K such that there does not exist a computable equivalence structure $\mathcal{B}$ with character K and with finitely many infinite equivalence classes. Then $\mathcal{A}$ is not computably categorical.*

Proof: If $\mathcal{A} = (\omega, \equiv_{\mathcal{A}})$ is computably categorical, then $\mathcal{C}_e \simeq \mathcal{A}$ if and only if $\mathcal{A}$ and $\mathcal{C}_e$ are computably isomorphic, which has the following Σ^0_3 definition:

$$(\exists a)[a \in Tot \ \& \ (\forall m)(\forall n)\,(\ m \equiv_e n \iff \phi_a(m) \equiv_{\mathcal{A}} \phi_a(n)\ )].$$

This contradicts the Π^0_3 completeness in Theorem 3.9. □

For characters with computable s_1-functions, a structure may have finitely many or infinitely many infinite equivalence classes, and there is a higher complexity of the index set.

Theorem 3.11 *Let $\mathcal{A}$ be a computable equivalence structure with unbounded character K and with a finite number r of infinite equivalence classes.*

(a) *If $r = 0$, then the index set $\{e : \mathcal{C}_e \simeq \mathcal{A}\}$ is Π^0_3 complete.*

(b) *If $r > 0$, then the index set $\{e : \mathcal{C}_e \simeq \mathcal{A}\}$ is D^0_3 complete.*

Proof: (a) Suppose that $\mathcal{A}$ has no infinite equivalence classes. Then $\{e : \mathcal{C}_e \simeq \mathcal{A}\}$ is a Π^0_3 set, since $\mathcal{C}_e \simeq \mathcal{A}$ if and only if the following two facts hold:

(1) $\chi(\mathcal{C}_e) = K$ (which is a Π^0_3 condition by Lemma 3.8);

(2) $\mathcal{C}_e$ has no infinite equivalence classes (which is a Π^0_3 condition by Lemma 3.7).

For the completeness, let P be a given Π^0_3 set. We construct a reduction of P to our index set as follows. Let g be a computable s_1-function for K, let $m_i = lim_s g(i, s)$, and let

$$f_0(i, s) = g(2i, s),$$

and

$$f_1(i, s) = g(2i + 1, s),$$

Hence both f_0 and f_1 are s_1-functions. Now let

$$K_1 = K \setminus \{\langle m_{2i}, n\} : i, n \in \omega\}.$$

Let ϕ be given by the proof of Lemma 3.8, so that $K_{\phi(e)} = K_1$ if and only if $e \in P$, and such that $K_{\phi(e)} \subseteq K_1$ for all e. Then the character

$$K_{\phi(e)} \cup (K \cap \{\langle m_{2i}, n\rangle : i, n \in \omega\})$$

has a computable s_1-function, so we can apply the proof of Lemma 2.8 to construct $\mathcal{C}_{\psi(e)}$ with character

$$K_{\phi(e)} \cup (K \cap \{\langle m_{2i}, n\rangle : i, n \in \omega\}),$$

which has no infinite equivalence classes. It is now clear that $e \in P$ if and only if $K_{\phi(e)} = K_1$, which is if and only if $C_{\psi(e)} \simeq \mathcal{A}$.

Note that if we simply apply Lemma 3.8 to K itself, we find that $K_{\phi(e)}$ is finite whenever $K_{\phi(e)} \neq K$, so that $\mathcal{C}_{\phi(e)}$ would also be finite, whereas we are assuming that all computable equivalence structures $\mathcal{C}_i$ have universe ω.

(b) Suppose that $\mathcal{A}$ has exactly $r > 0$ infinite equivalence classes. Then $\{e : C_e \simeq \mathcal{A}\}$ is a D^0_3 set (the difference of two Σ^0_3 sets), since $\mathcal{C}_e \simeq \mathcal{A}$ if and only if the following two facts hold:

(1) $\chi(\mathcal{C}_e) = K$ (which is a Π^0_3 condition by Lemma 2.2(c) or Lemma 3.8);

(2) $\mathcal{C}_e$ has exactly r infinite equivalence classes (which is a D^0_3 condition by Lemma 3.7).

For the completeness, let P be a Π^0_3 set as in part (a), and let Q be a Σ^0_3 set. Now let R be a Π^0_2 set such that for all d,

$$d \in Q \iff (\exists c)(\langle c, d\rangle \in R).$$

Without loss of generality, we may assume that when $d \in Q$, then there exists a unique c such that $\langle c, d\rangle \in R$. It follows from the Π^0_2 completeness of $\{e : W_e \text{ is infinite}\}$ that there is a computable set T such that

$$\langle c, d\rangle \in R \iff (\{t : \langle c, d, t\rangle \in T\} \text{ is infinite}).$$

We will define a computable function θ so that for all d and e,

$$\mathcal{C}_{\theta(d,e)} \simeq \mathcal{A} \iff d \in Q \;\&\; e \in P.$$

The structure $\mathcal{C}_{\theta(d,e)}$ will be the disjoint union of three components.

The first component will be a structure $\mathcal{B}$ that has no infinite equivalence classes and has character $K_{\phi(e)}$, where

$$e \in P \iff K_{\phi(e)} = K_1.$$

This is constructed as in part (a).

The second component $\mathcal{C}$ is fixed for all e, has no infinite equivalence classes, and has character

$$\{\langle m_{2i}, n\rangle : n > 0 \;\&\; \langle m_{2i}, n+1\rangle \in K\}.$$

This might be a finite structure.

The third component $\mathcal{D}$ will have character $\{\langle m_{2i}, 1\rangle : i \in \omega\}$, and will have exactly r infinite equivalence classes if $d \in Q$, and no infinite equivalence classes if $d \notin Q$. We give the proof for $r = 1$ and leave the general case to the reader.

From the s_1-function f_0 we create an infinite set of s_1-functions g_c, where

$$g_c(i, s) = f_0(2^c(2i + 1), s).$$

Let

$$m_{c,i} = lim_s g_c(i, s).$$

Then $\mathcal{D}$ will be the disjoint union of equivalence structures $\mathcal{D}_c$ having character

$$K \cap \{\langle m_{c,i}, n\rangle : i, n \in \omega\},$$

and having exactly one infinite equivalence class if $\langle c, d\rangle \in Q$, and no infinite equivalence classes otherwise. It now suffices to construct $\mathcal{D}_c$ with universe ω.

Fix c and let $n_i = m_{c,i}$. The construction of the equivalence relation E in $\mathcal{D}_c$ is in stages. At stage s, there will be equivalence classes C_i^s of sizes $g_c(i,s)$ for all $i < s$. There will be a particular $i = i^s$ such that

$$\{2t : \langle c, d, t\rangle \in T\} \subseteq C_i^s.$$

This class C_i^s is the *test* class. Initially, we have the empty structure. By stage s, all numbers $< s$ will have been assigned to an equivalence class, and hence we will have decided whether aEb for all $a, b < 2s$.

At *stage* $t+1$, let $i = i^t$ and check whether $\langle c, d, t\rangle \in T$. If it is true, then we let $i^{t+1} = t$, and create the new class C_t^{t+1} by adding to C_i^t the element $2t$, along with $g(t, t+1) - g(i, t) - 1$ new odd numbers. We also create a new class C_i^{t+1} with $g(i, t+1)$ new odd numbers. For all j such that $j < t$ and $j \neq i$, we add $g_c(j, t+1) - g_c(j, t)$ odd numbers to the class C_j^t to obtain C_j^{t+1}.

If $\langle c, d, t\rangle \notin T$, then for all $j < t$, we simply add $g_c(j, t+1) - g_c(j, t)$ odd numbers to the class C_j^t to obtain C_j^{t+1}, and we create the new class C^t with exactly $g_c(t, t+1)$ new odd elements.

There are two possible outcomes of this construction. If $\{t : \langle c, d, t\rangle \in T\}$ is finite, then after some stage t, i^t becomes fixed, and thus has limit i. Then for every i, the class $C_i = \cup_t C_i^t$ will have exactly n_i elements, and every number will belong to one of these classes. Thus, $\mathcal{D}_c$ has character $\{\langle n_i, 1\rangle : i \in \omega\}$ and has no infinite equivalence classes. If $\{t : \langle c, d, t\rangle \in T\}$ is infinite, then $lim_t i^t = \infty$ and $\mathcal{D}_c$ has one additional, infinite equivalence class, the test class, which is $\cup_t C_{i^t}^t$.

It follows that if $d \notin Q$, then each $\mathcal{D}_c$ has character $\{\langle m_{c,i}, 1\rangle : i \in \omega\}$ and has no infinite equivalence classes, so that $\mathcal{D}$ has character $\{\langle m_{2i}, 1\rangle : i \in \omega\}$ and has no infinite equivalence classes. If $d \in Q$, then one of the $\mathcal{D}_c$ has one infinite equivalence class, and the others have no infinite equivalence classes. Hence $\mathcal{D}$ has exactly one infinite equivalence class, as desired. □

Theorem 3.12 *Let K be an unbounded Σ_2^0 character, and let $\mathcal{A}$ be a computable equivalence structure with character K and with finitely many infinite equivalence classes. Then $\mathcal{A}$ is not computably categorical.*

Proof: This follows immediately from Theorem 3.11 as in the proof of Theorem 3.10. □

Note that for the structure $\mathcal{A}$ of Theorem 3.12, and the corresponding structure $\mathcal{B}$ isomorphic but not computably isomorphic to $\mathcal{A}$, both $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$ are computable, since there are finitely many infinite equivalence classes.

Theorem 3.13 *Let $\mathcal{A}$ be an equivalence structure with unbounded character K and with infinitely many infinite equivalence classes. Suppose that there exists an equivalence structure $\mathcal{B}$ with character K and with finitely many infinite equivalence classes. Then the index set $\{e : \mathcal{C}_e \simeq \mathcal{A}\}$ is Π_4^0 complete.*

Proof: It follows from Lemma 2.6 that K possesses a computable s_1-function g. Let $m_i = lim_s g(i,s)$. Let $f_0(i,s) = g(i,2s)$ and $f_1(i,s) = g(i,2s+1)$, so that both f_0 and f_1 are s_1-functions. From the s_1-function f_0 we create an infinite set of s_1-functions $g_c(i,s)$, where

$$g_c(i,s) = f_0(2^c(2i+1),s).$$

Set $m_{c,i} = lim_s g_c(i,s)$. Let

$$K_0 = K \setminus \{\langle m_{2i}, n\} : i, n \in \omega\},$$

and for each c, let

$$K_{c+1} = K \cap \{\langle m_{c,i}, n\rangle : i, n \in \omega\}.$$

Thus, K is the disjoint union of the characters K_c. Now, K_0 has s_1-function f_1, and, therefore, there is a structure $\mathcal{A}_0$ with character K_0 and no infinite equivalence classes.

Let P be a Π^0_4 set. Let Q be a Σ^0_3 relation such that

$$e \in P \iff (\forall c)Q(e,c).$$

We may assume that if $e \notin P$, then $\{c : Q(e,c)\}$ is finite. By uniformizing the proof of Theorem 3.11, we obtain that there is a computable binary function ϕ such that $\mathcal{C}_{\phi(e,c)}$ has character K_{c+1} for all e and c, and has exactly one infinite equivalence class if $Q(e,c)$, and no infinite equivalence classes if $\neg Q(e,c)$.

Now define $\mathcal{C}_{\psi(e)} = (\omega, E)$ as the effective union of the structures $\mathcal{A}_0, \mathcal{C}_{\phi(e,c)}$. That is, let E_0 be the equivalence relation of $\mathcal{A}_0$, and let $E_{e,c}$ be the equivalence relation of $\mathcal{C}_{\phi(e,c)}$. Let

$$E(2a, 2b) \iff E_0(a,b),$$

and for each c, let

$$E(2^c(2a+1), 2^c(2b+1)) \iff E_{e,c}(a,b),$$

while for all other i, j, we let $\neg E(i,j)$. Then, clearly, the structure $\mathcal{C} = (\omega, E)$ has character $K = \cup_c K_c$.

If $e \in P$, then $Q(e,c)$ holds for all c, so each $\mathcal{C}_{\phi(e,c)}$ has an infinite equivalence class. Hence $\mathcal{C}_{\psi(e)}$ has infinitely many infinite equivalence classes.

If $e \notin P$, then $Q(e,c)$ holds for finitely many c, so finitely many $\mathcal{C}_{\phi(e,c)}$ have exactly one infinite equivalence class, and the others have no infinite equivalence classes. Thus, $\mathcal{C}_{\psi(e)}$ has finitely many infinite equivalence classes.

It follows that

$$\mathcal{C}_{\psi(e)} \simeq \mathcal{A} \Longleftrightarrow e \in P.$$

□

Theorem 3.14 *Let $\mathcal{A}$ be a computable equivalence structure with unbounded character K and with infinitely many infinite equivalence classes, such that there exists a computable equivalence structure $\mathcal{B}$ with character K and with finitely many infinite equivalence classes. Then $\mathcal{A}$ is not Δ^0_2 categorical.*

Proof: If $\mathcal{A} = (\omega, \equiv_{\mathcal{A}})$ is Δ_2^0 categorical, then $\mathcal{C}_e \simeq \mathcal{A}$ if and only if $\mathcal{A}$ and $\mathcal{C}_e$ are Δ_2^0 isomorphic. Thus the set $\{e : \mathcal{C}_e \simeq \mathcal{A}\}$ has a Σ_4^0 definition. That is, with a c.e. complete set M as an oracle, we have

$$(\exists a)[a \in Tot^M \& (\forall m)(\forall n)(m \equiv_e n \Longleftrightarrow \phi_a^M(m) \equiv_{\mathcal{A}} \phi_a^M(n))].$$

This contradicts the Π_4^0 completeness in Theorem 3.13. □

Combining these results, we obtain the following corollary.

Corollary 3.15 *No equivalence structure with unbounded character is computably categorical.*

We can now establish that for computable equivalence structures computable categoricity and relative computable categoricity coincide.

Theorem 3.16 *All computably categorical equivalence structures are also relatively computably categorical.*

Proof: Suppose that $\mathcal{A}$ is not relatively computably categorical and has character K. It follows from Propositions 3.1 and 3.2 that $\mathcal{A}$ has infinitely many finite equivalence classes. First, suppose that K is bounded. Then there exists finite k such that $\mathcal{A}$ has infinitely many classes of size k. It now follows from Proposition 3.2 that either $\mathcal{A}$ has infinitely many infinite classes, or there are two distinct finite numbers k_1 and k_2 such that $\mathcal{A}$ has infinitely many classes of size k_1 and infinitely many classes of size k_2. In either case, Theorem 3.5 implies that $\mathcal{A}$ is not computably categorical.

Now, suppose that K is unbounded. Then there are two possibilities. Suppose first that K has no computable s_1-function. Then, by Theorem 3.10, $\mathcal{A}$ is not computably categorical. Next, suppose that K has a computable s_1-function, and that $\mathcal{A}$ has infinitely many infinite equivalence classes. Then, by Theorem 3.14, $\mathcal{A}$ is not computably categorical. Finally, suppose that $\mathcal{A}$ has only finitely many infinite equivalence classes. Then $\mathcal{A}$ is not computably categorical by Theorem 3.12. □

4 Δ_2^0 categoricity of equivalence structures

We continue with the analysis of Δ_2^0 categoricity. We already have, by Theorem 3.14, the following result.

Theorem 4.1 *Let $\mathcal{A}$ be a computable equivalence structure with infinitely many infinite equivalence classes and with unbounded character, which has a computable s_1-function. Then $\mathcal{A}$ is not Δ_2^0 categorical.*

We now consider equivalence structures with bounded character.

Theorem 4.2 *If $\mathcal{A}$ is a computable equivalence structure with bounded character, then $\mathcal{A}$ is relatively Δ_2^0 categorical.*

Proof: Let k be the maximum size of any finite equivalence class. The key fact here is that $[a]$ is infinite if and only if $[a]$ contains at least $k+1$ elements, which is a Σ_1^0 condition. By Lemma 2.2, there is a Δ_2^0 formula which characterizes the elements a with a finite equivalence class of size m. Then a Scott formula for the tuple $(a_1, \ldots, a_m)$ includes a formula $\psi_i(x_i)$ for each a_i, giving the cardinality of $[a_i]$, together with formulas $\psi_{i,j}(x_i, x_j)$ for each i, j, which express whether $a_i E^{\mathcal{A}} a_j$ and whether $a_i = a_j$. It now follows, as in the proof of Proposition 3.1, that whenever $\overrightarrow{a}$ and $\overrightarrow{b}$ have the same Scott formula, then there is an automorphism of $\mathcal{A}$ taking $\overrightarrow{a}$ to $\overrightarrow{b}$. Thus, $\mathcal{A}$ is relatively Δ_2^0 categorical. □

Theorem 4.3 *If $\mathcal{A}$ is a computable equivalence structure with finitely many infinite equivalence classes, then $\mathcal{A}$ is relatively Δ_2^0 categorical.*

Proof: By the proof of Proposition 3.2, there is a Σ_1^0 Scott formula for each element with an infinite equivalence class. There is a Σ_2^0 Scott formula for each element with a finite equivalence class, by the proof of Theorem 4.2. It now follows, as before, that $\mathcal{A}$ is relatively Δ_2^0 categorical. □

The previous result leads to a stronger result for structures $\mathcal{A}$ with $Fin^{\mathcal{A}}$ computable.

Theorem 4.4 *For any two isomorphic computable equivalence structures $\mathcal{A}$ and $\mathcal{B}$ such that $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$ are both computable, $\mathcal{A}$ and $\mathcal{B}$ are Δ_2^0 isomorphic.*

Proof: It follows from Proposition 3.1 that $Inf^{\mathcal{A}}$ and $Inf^{\mathcal{B}}$ are computably isomorphic, and it follows from Theorem 4.3 that $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$ are Δ_2^0 isomorphic. Now, the two corresponding isomorphisms may be combined into a Δ_2^0 isomorphism between $\mathcal{A}$ and $\mathcal{B}$, since $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$ are computable.

In fact, we observe that this result still holds if we only assume that $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$ are both Δ_2^0. The substructures $Inf^{\mathcal{A}}$ and $Inf^{\mathcal{B}}$ are still Δ_2^0 isomorphic, and there is a Δ_2^0 enumeration of the finite equivalence classes of $\mathcal{A}$ and $\mathcal{B}$, which will induce a Δ_2^0 isomorphism between $Fin^{\mathcal{A}}$ and $Fin^{\mathcal{B}}$. □

It remains to consider the case of an unbounded character K with no computable s_1-function. Recall from Lemma 2.3 that we may construct a computable equivalence structure $\mathcal{A}$ with $Fin^{\mathcal{A}}$ being a Π_1^0 set. If we could also construct a computable equivalence structure $\mathcal{B}$ such that $Fin^{\mathcal{B}}$ is not a Δ_2^0 set, then it would follow that $\mathcal{B}$ is not Δ_2^0 isomorphic to $\mathcal{A}$. Surprisingly, we cannot make $Fin^{\mathcal{B}}$ a complete Σ_2^0 set, as we could when K had a computable s_1-function. This is because of the following result.

Theorem 4.5 *Let $\mathcal{A}$ be a computable equivalence structure, and let C be an infinite c.e. subset of $Fin^{\mathcal{A}}$. Then there is a computable equivalence structure $\mathcal{A}_1$ with character $K_1 \subseteq \chi(\mathcal{A})$ and no infinite equivalence classes. Furthermore,* *if* $\{card([c]) : c \in C\}$ *is unbounded, then* $\mathcal{A}$ *possesses a computable* s_1*-function* f*. Thus, there is a computable structure with character* $\chi(\mathcal{A})$ *and with no infinite equivalence classes.*

Proof: Let $\mathcal{A} = (\omega, E)$, where E is a computable equivalence relation. Let

$$C_1 = \{a : (\exists c \in C) cEa\}.$$

Then C_1 is an infinite c.e. set. Fix a computable 1-1 enumeration $\{c_0, c_1, \dots\}$ of C_1. Now let

$$f(i, s) = card(\{x \leq s : xEc_i\}).$$

Let $\mathcal{A}_1 = (\omega, E_1)$, where iE_1j if and only if c_iEc_j. Let $K_1 = \chi(\mathcal{A}_1)$. Then f is clearly an s-function for K_1, and it follows from Lemma 2.8 that there is a computable structure with character K_1 and with no infinite equivalence classes. Clearly,

$$K_1 \subset \chi(\mathcal{A}).$$

Now suppose that $\{card([c]) : c \in C\}$ is unbounded. Then $\mathcal{A}_1$ has unbounded character and no infinite equivalence classes. Thus, by Lemma 2.6, K_1 has a computable s_1-function, and hence $\chi(\mathcal{A})$ has the same s_1-function. It now follows from Lemma 2.8 that there is a computable structure with character $\chi(\mathcal{A})$ and with no infinite equivalence classes. □

Theorem 4.6 *Let* K *be an unbounded character. If* K *has no computable* s_1*-function, then there is no computable equivalence structure* $\mathcal{A}$ *with character* K *such that* $Fin^{\mathcal{A}}$ *is* Σ^0_2 *complete, or even* Σ^0_1 *hard.*

Proof: Let M be a complete c.e. set, and suppose that there were a computable function f such that

$$i \in M \iff f(i) \in Fin^{\mathcal{A}}.$$

Then $C = \{f(i) : i \in M\}$ is a c.e. subset of $Fin^{\mathcal{A}}$. If C is finite, say $C = \{c_1, \dots, c_t\}$, then

$$i \in M \iff (\ f(i) = c_1 \vee f(i) = c_2 \vee \cdots \vee f(i) = c_t\ ),$$

so M is a computable set. Thus, C is infinite. Now suppose that $\{card([c]) : c \in C\}$ is bounded by some finite k. Then C is a subset of the Π^0_1 set P, where

$$P = \{a : card([a]) \leq k\}.$$

Since we have

$$i \in M \iff f(i) \in P,$$

that would imply that M is a Π^0_1 set. This contradiction shows that $\{card([c]) : c \in C\}$ is unbounded. It now follows by Theorem 4.5 that K possesses a computable s_1-function. □

Open Question: Let $\mathcal{A}$ be a computable equivalence structure having unbounded character, infinitely many infinite equivalence classes, and no computable s_1-function. Furthermore, assume that $Fin^{\mathcal{A}}$ is Turing incomparable with $\emptyset'$. Does such a structure exist, and if so, is this structure Δ_2^0 categorical?

Theorem 4.6 may provide some evidence that such a structure, if it exists, could in fact be Δ_2^0 categorical. Nevertheless, we can show that such a structure cannot be relatively Δ_2^0 categorical.

Theorem 4.7 *Let $\mathcal{A}$ be a computable equivalence structure. If $\mathcal{A}$ has unbounded character and infinitely many infinite equivalence classes, then $\mathcal{A}$ is not relatively Δ_2^0 categorical.*

Proof: Suppose, on the contrary, that an element a with an infinite equivalence class had a Σ_2 Scott formula $\psi(x, \overrightarrow{d})$. Since there are only finitely many parameters $\overrightarrow{d}$ involved, we may assume that $[a]$ does not contain any of the parameters $\overrightarrow{d}$. (This is where we use the assumption that there are infinitely many infinite equivalence classes.) Then, by choosing elements $c_1, \dots, c_n$ of $\mathcal{A}$ to instantiate the existentially quantified variables in $\psi(x, \overrightarrow{d})$, we would have a computable Π_1 formula $\theta(x, \overrightarrow{d}, \overrightarrow{c})$ satisfied by a, where $\overrightarrow{c} = c_1, \dots, c_n$.

Now, it is easy to see that for any submodel $\mathcal{M}$ of $\mathcal{A}$, which contains a, $\overrightarrow{d}$ and $\overrightarrow{c}$, we have $\mathcal{M} \models \theta(a, \overrightarrow{d}, \overrightarrow{c})$. In fact, since our structures are relational, $\mathcal{A} \models \theta(b, \overrightarrow{d}, \overrightarrow{e})$ if and only if $\mathcal{M} \models \theta(b, \overrightarrow{d}, \overrightarrow{e})$ for all finite submodels $\mathcal{M}$ of $\mathcal{A}$, which contain b, $\overrightarrow{d}$ and $\overrightarrow{e}$.

Thus, in particular, for the finite subset $C = \{a\} \cup \{\overrightarrow{c}\} \cup \{\overrightarrow{d}\}$ of ω, we have $\mathcal{C} = (C, E^{\mathcal{C}}) \models \theta(a, \overrightarrow{d}, \overrightarrow{c})$. Suppose that $\overrightarrow{c}$ contains $m \leq n$ elements of $[a]$, and choose b such that $[b] \cap C = \emptyset$ and $m < card([b]) < \omega$. (Here we use the fact that the character of $\mathcal{A}$ is unbounded.) Let $\mathcal{B}$, where $\mathcal{B} \simeq \mathcal{C}$, contain m elements of $[b]$ (including b), together with $C \setminus [a]$. Let $\overrightarrow{e}$ denote the image of $\overrightarrow{c}$ under the isomorphism between $\mathcal{C}$ and $\mathcal{B}$. Then $\mathcal{B} \models \theta(b, \overrightarrow{d}, \overrightarrow{e})$. Furthermore, let $\mathcal{B}'$ be any finite submodel of $\mathcal{A}$ such that $\mathcal{B} \subseteq \mathcal{B}'$. Then it is easy to extend $\mathcal{C}$ to a finite submodel $\mathcal{C}'$ that is isomorphic to $\mathcal{B}'$ (where the isomorphism fixes $\overrightarrow{d}$ pointwise and takes a to b). Thus, $\mathcal{B}' \models \theta(b, \overrightarrow{d}, \overrightarrow{e})$ as well. It follows that $\mathcal{A} \models \theta(b, \overrightarrow{d}, \overrightarrow{e})$. Hence $\mathcal{A} \models \psi(b, \overrightarrow{d})$.

But there certainly can be no automorphism of $\mathcal{A}$ mapping a to b, since $[a]$ is infinite and $[b]$ is finite. Thus, in fact, a cannot have a Σ_2 Scott formula. $\square$

Corollary 4.8 *A computable equivalence structure $\mathcal{A}$ is relatively Δ_2^0 categorical if and only if $\mathcal{A}$ has finitely many infinite equivalence classes or $\mathcal{A}$ has bounded character.*

We conclude this section by looking at Δ_3^0 categoricity.

Theorem 4.9 *Every computable equivalence structure is relatively Δ^0_3 categorical.*

Proof: Any element with an infinite equivalence class has a Π_2 Scott formula, while the other elements even have Δ_2 Scott formulas. Thus, every tuple $(a_1, \ldots, a_n)$ has a Σ_3 Scott formula. □

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