Colored $\mathfrak{sl}(N)$ link homology via matrix factorizations

Hao Wu

George Washington University

Overview

The Reshetikhin-Turaev $\mathfrak{sl}(N)$ polynomial of links colored by wedge powers of the defining representation has been categorified via several different approaches.

I'll talk about the categorification using matrix factorizations, which is a direct generalization of the Khovanov-Rozansky homology.

I'll also also review deformations and applications of this categorification.

Abstract MOY graphs

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An abstract MOY graph is called trivalent is all of its internal vertices have valence 3.

Embedded MOY graphs

An embedded MOY graph, or simply a MOY graph, Γ is an embedding of an abstract MOY graph into $\mathbb{R}^2$ such that, through each vertex v of Γ, there is a straight line L_v so that all the edges entering v enter through one side of L_v and all edges leaving v leave through the other side of L_v.

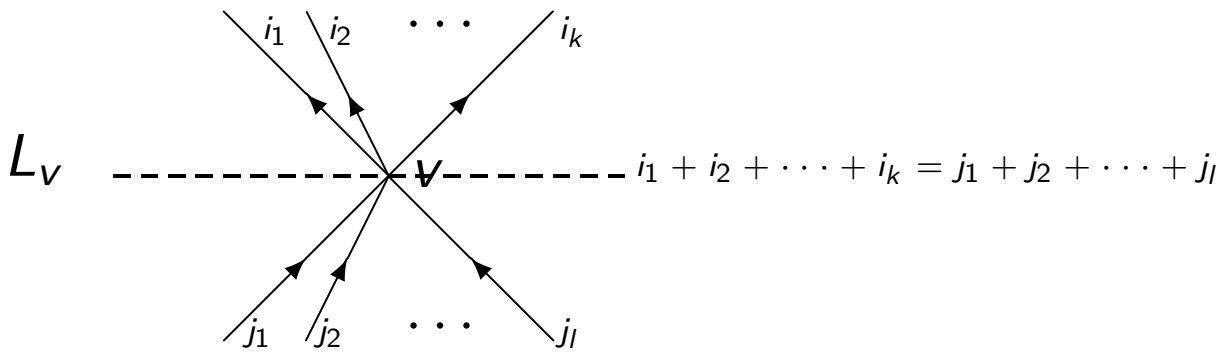
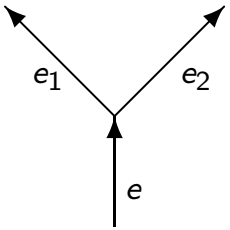


Figure: An internal vertex of a MOY graph

Trivalent MOY graphs and their states



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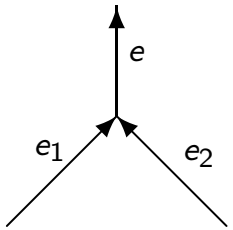

Let Γ be a closed trivalent MOY graph, and $E(\Gamma)$ the set of edges of Γ. Denote by $\mathrm{c} : E(\Gamma) \to \mathbb{N}$ the color function of Γ. That is, for every edge e of Γ, $\mathrm{c}(e) \in \mathbb{N}$ is the color of e.

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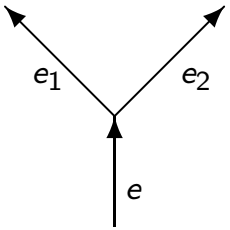

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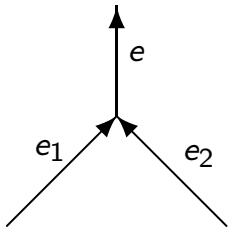

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Define $\mathcal{N} = \{-N+1, -N+3, \cdots, N-3, N-1\}$ and $\mathcal{P}(\mathcal{N})$ to be the set of subsets of $\mathcal{N}$.

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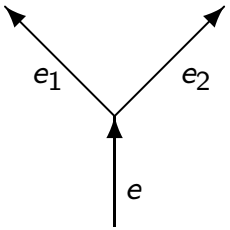

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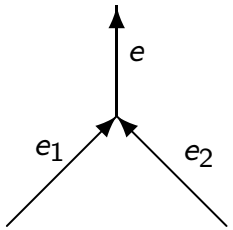

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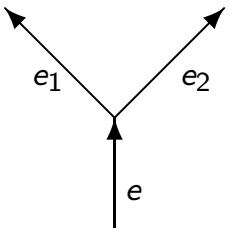
A state of Γ is a function $\sigma : E(\Gamma) \to \mathcal{P}(\mathcal{N})$ such that

(i) For every edge e of Γ, $\#\sigma(e) = \mathrm{c}(e)$.

(ii) For every vertex v of Γ, as depicted above, we have $\sigma(e) = \sigma(e_1) \cup \sigma(e_2)$. (In particular, this implies that $\sigma(e_1) \cap \sigma(e_2) = \emptyset$.)

Weight



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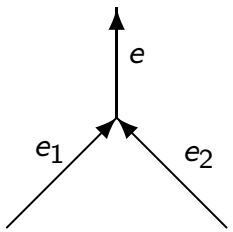

For a state σ of Γ and a vertex v of Γ as depicted above, the weight of v with respect to σ is defined to be

$$\mathrm{wt}(v;\sigma) = q^{\frac{\mathrm{c}(e_1)\mathrm{c}(e_2)}{2} - \pi(\sigma(e_1),\sigma(e_2))},$$

where $\pi : \mathcal{P}(\mathcal{N}) \times \mathcal{P}(\mathcal{N}) \to \mathbb{Z}_{\geq 0}$ is define by

$$\pi(A_1, A_2) = \#\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 > a_2\} \text{ for } A_1,\ A_2 \in \mathcal{P}(\mathcal{N}).$$

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This changes Γ into a collection $\{C_1, \ldots, C_k\}$ of embedded oriented circles, each of which is assigned an element $\sigma(C_i)$ of $\mathcal{N}$.

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The rotation number $\mathrm{rot}(\sigma)$ of σ is then defined to be

$$\mathrm{rot}(\sigma) = \sum_{i=1}^{k} \sigma(C_i)\mathrm{rot}(C_i).$$

The $\mathfrak{sl}(N)$ MOY graph polynomial

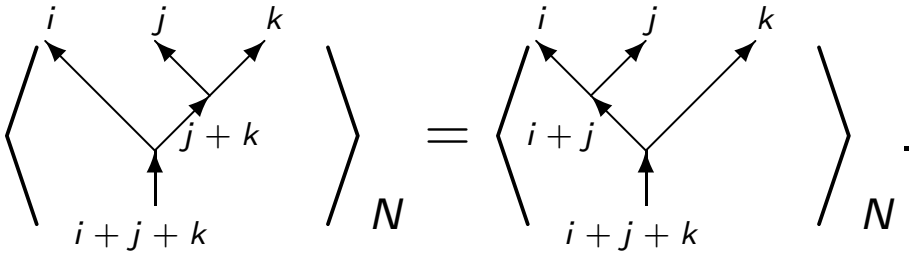
The $\mathfrak{sl}(N)$ MOY polynomial of Γ is defined to be

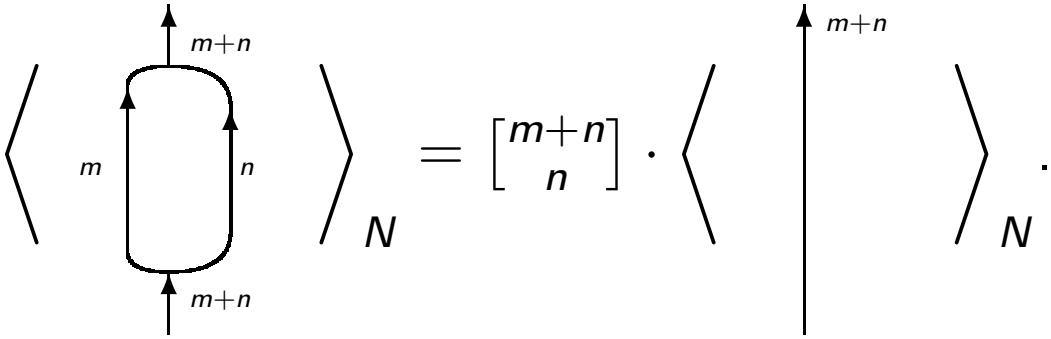
$$\langle \Gamma \rangle_N := \sum_{\sigma} (\prod_{v} \mathrm{wt}(v;\sigma)) q^{\mathrm{rot}(\sigma)},$$

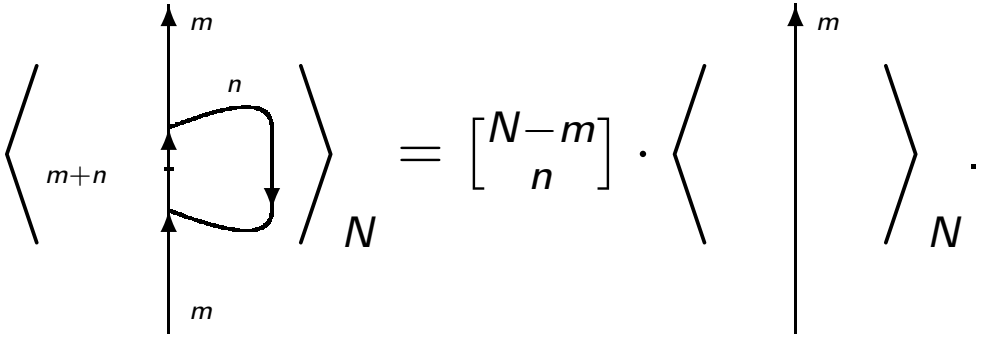
where σ runs through all states of Γ and v runs through all vertices of Γ.

MOY relations (1–4)

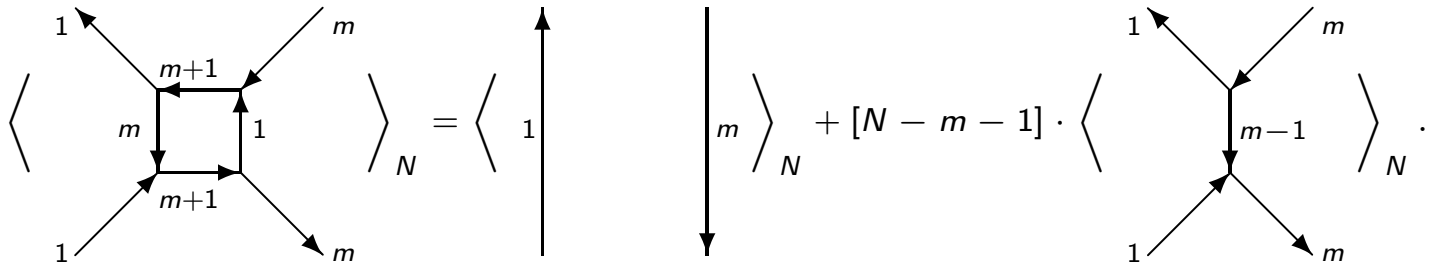
1. $\langle \bigcirc_m \rangle_N = \begin{bmatrix} N \\ m \end{bmatrix}$, where $\bigcirc_m$ is a circle colored by m.

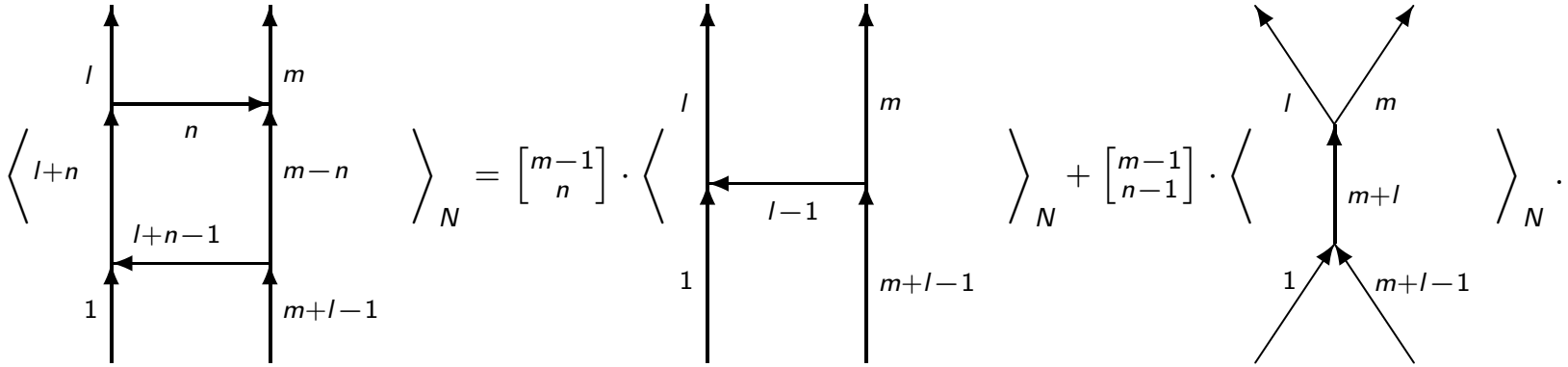
2. 

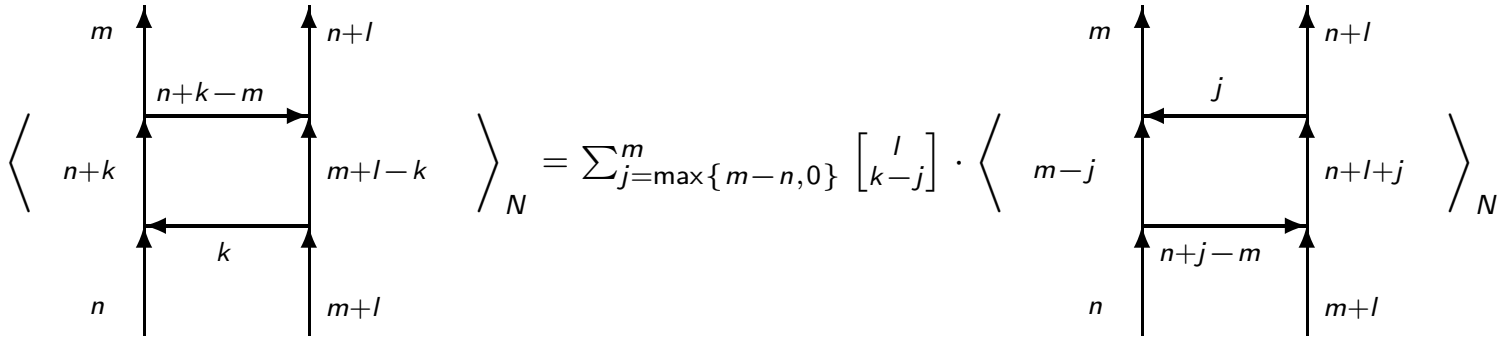
3. 

4. 

MOY relations (5–7)

5. 

6. 

7. 

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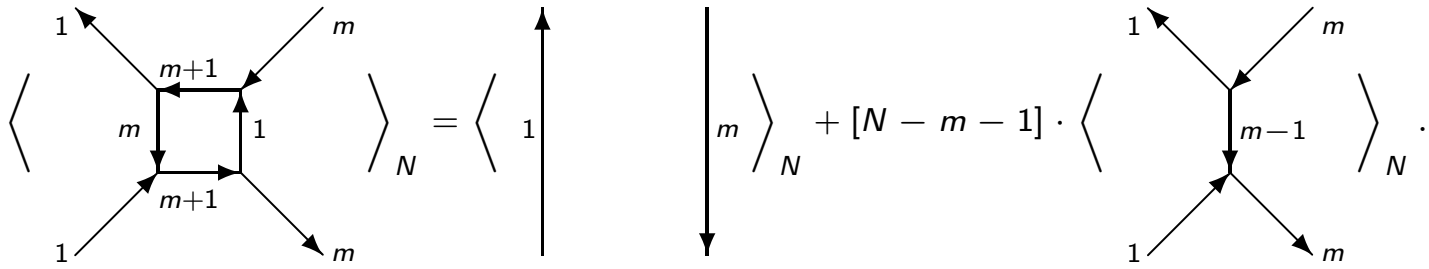
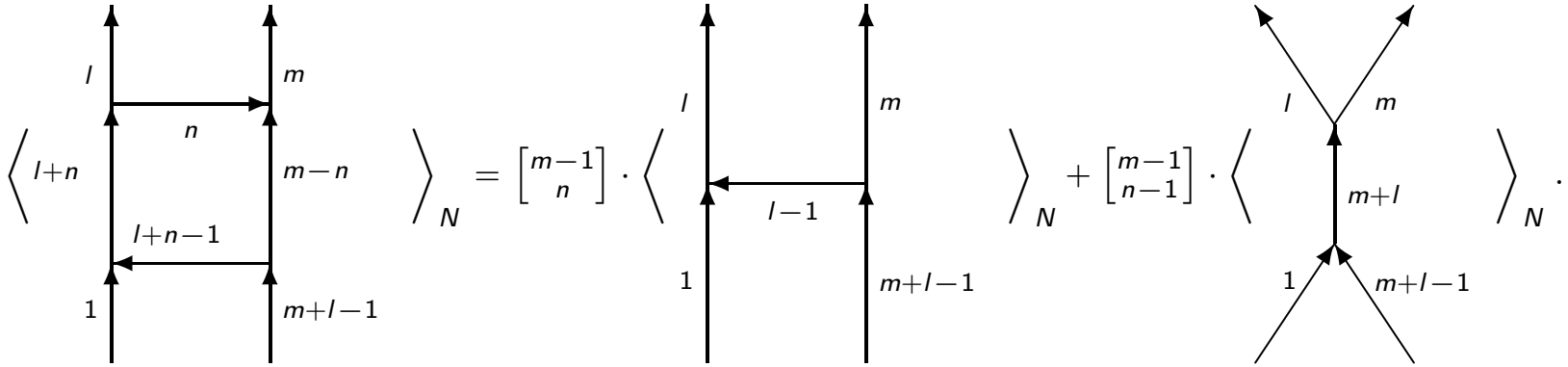
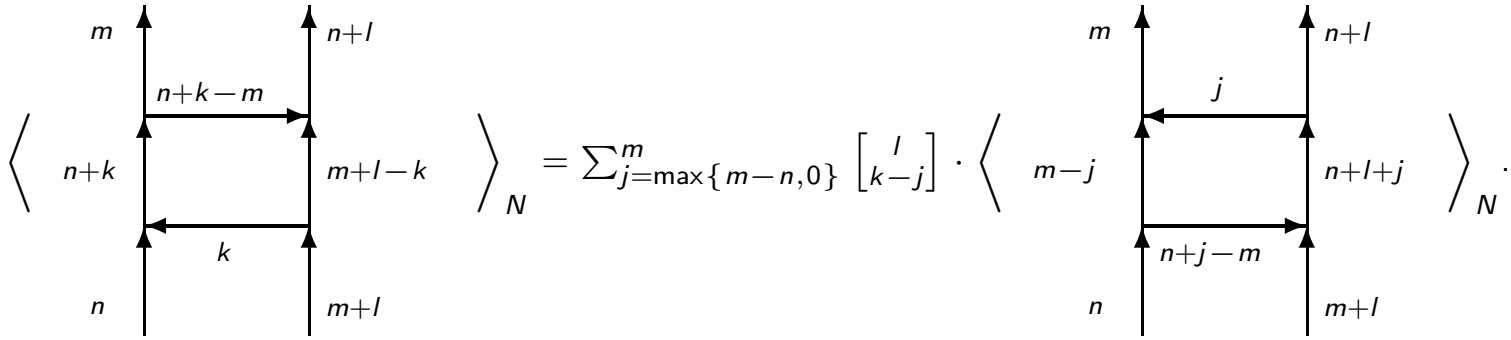
5. 

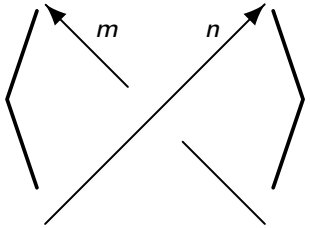
6. 

7. 

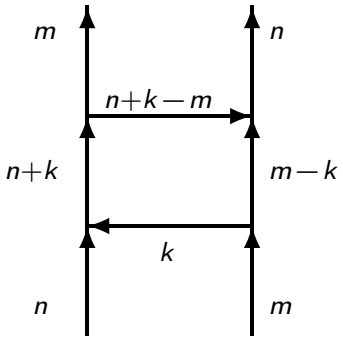
The above MOY relations uniquely determine the $\mathfrak{sl}(N)$ MOY graph polynomial.

Unnormalized colored Reshetikhin-Turaev $\mathfrak{sl}(N)$ polynomial

For a link diagram D colored by non-negative integers, define $\langle D \rangle_N$ by applying the following at every crossing of D.



$$\Big\rangle_N = \sum_{k=\max\{0,m-n\}}^{m} (-1)^{m-k} q^{k-m} \Big\langle$$



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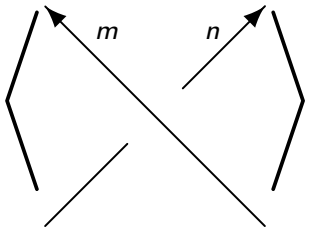

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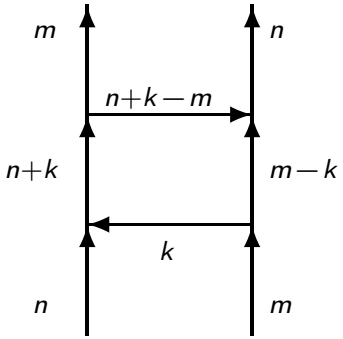
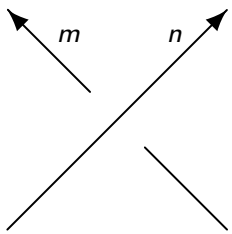
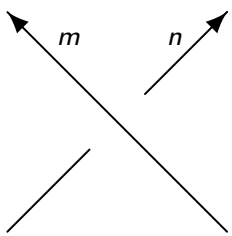


$$\Big\rangle_N .$$

Normalized colored Reshetikhin-Turaev $\mathfrak{sl}(N)$ polynomial

For each crossing c of D, define the shifting factor $\mathrm{s}(c)$ of c by

$\mathrm{s}\Bigg($  $\Bigg) = \begin{cases} (-1)^{-m} q^{m(N+1-m)} & \text{if } m = n, \\ 1 & \text{if } m \neq n, \end{cases}$

$\mathrm{s}\Bigg($  $\Bigg) = \begin{cases} (-1)^{m} q^{-m(N+1-m)} & \text{if } m = n, \\ 1 & \text{if } m \neq n. \end{cases}$

The normalized Reshetikhin-Turaev $\mathfrak{sl}(N)$-polynomial $\mathrm{RT}_D(q)$ of D is

$$\mathrm{RT}_D(q) = \langle D \rangle_N \cdot \prod_c \mathrm{s}(c),$$

where c runs through all crossings of D.

Graded matrix factorizations

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A graded matrix factorization M over R with potential w is a collection of two graded free R-modules M_0, M_1 and two homogeneous R-module homomorphisms $d_0 : M_0 \to M_1$, $d_1 : M_1 \to M_0$ of degree $N + 1$, called differential maps, s.t.

$$d_1 \circ d_0 = w \cdot \mathrm{id}_{M_0}, \qquad d_0 \circ d_1 = w \cdot \mathrm{id}_{M_1}.$$

We usually write M as

$$M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0.$$

Koszul Matrix Factorizations

If $a_0, a_1 \in R$ are homogeneous s.t. $\deg a_0 + \deg a_1 = 2N + 2$, then denote by $(a_0, a_1)_R$ the graded matrix factorization

$$R \xrightarrow{a_0} R\{q^{N+1-\deg a_0}\} \xrightarrow{a_1} R,$$

which has potential $a_0 a_1$.

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which has potential $a_0 a_1$.
If $a_{1,0}, a_{1,1}, \ldots, a_{k,0}, a_{k,1} \in R$ are homogeneous with $\deg a_{j,0} + \deg a_{j,1} = 2N + 2$, then define

$$\begin{pmatrix} a_{1,0}, & a_{1,1} \\ a_{2,0}, & a_{2,1} \\ \cdots & \cdots \\ a_{k,0}, & a_{k,1} \end{pmatrix}_R$$

to be the tenser product

$$(a_{1,0}, a_{1,1})_R \otimes_R (a_{2,0}, a_{2,1})_R \otimes_R \cdots \otimes_R (a_{k,0}, a_{k,1})_R,$$

which is a graded matrix factorization with potential $\sum_{j=1}^{k} a_{j,0} \cdot a_{j,1}$.

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$$\text{elementary:} \qquad X_k := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} x_{i_1} x_{i_1} \cdots x_{i_k},$$

$$\text{complete:} \qquad h_k(\mathbb{X}) := \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le m} x_{i_1} x_{i_1} \cdots x_{i_k},$$

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\end{aligned}$$

$$\begin{aligned}
\mathrm{Sym}(\mathbb{X}) &= \mathbb{C}[X_1, \ldots, X_m] = \mathbb{C}[h_1(\mathbb{X}), \ldots, h_m(\mathbb{X})] \\
&= \mathbb{C}[p_1(\mathbb{X}), \ldots, p_m(\mathbb{X})]
\end{aligned}$$

Symmetric polynomials (cont'd)

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This is a graded-free module whose structure is known.

Markings of MOY graphs

A marking of an MOY graph Γ consists the following:

1. A finite collection of marked points on Γ such that
   - every edge of Γ has at least one marked point;
   - all the end points (vertices of valence 1) are marked;
   - none of the internal vertices (vertices of valence at least 2) is marked.

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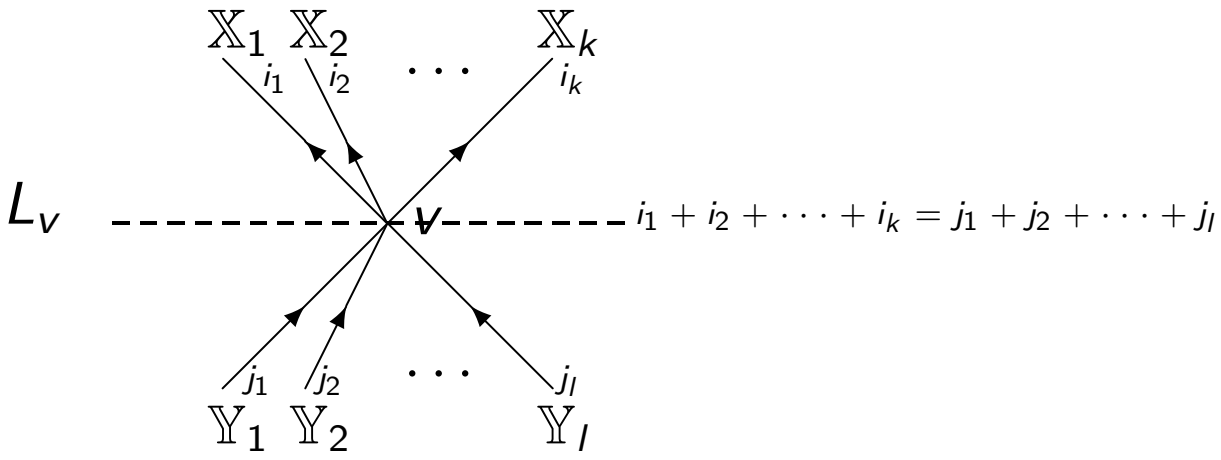
   - none of the internal vertices (vertices of valence at least 2) is marked.
2. An assignment of pairwise disjoint alphabets to the marked points such that the alphabet associated to a marked point on an edge of color m has m independent indeterminates.

Matrix factorization associated to a vertex



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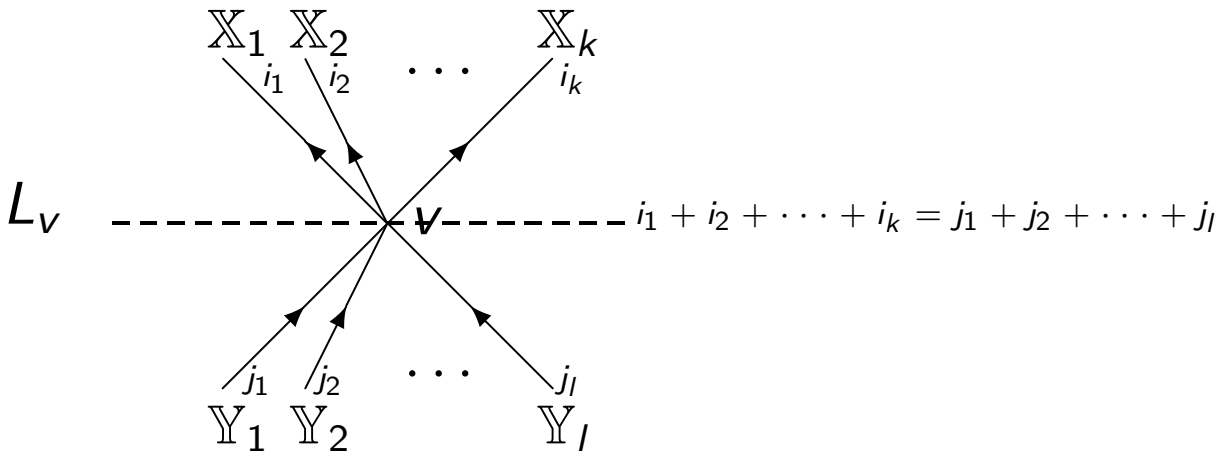
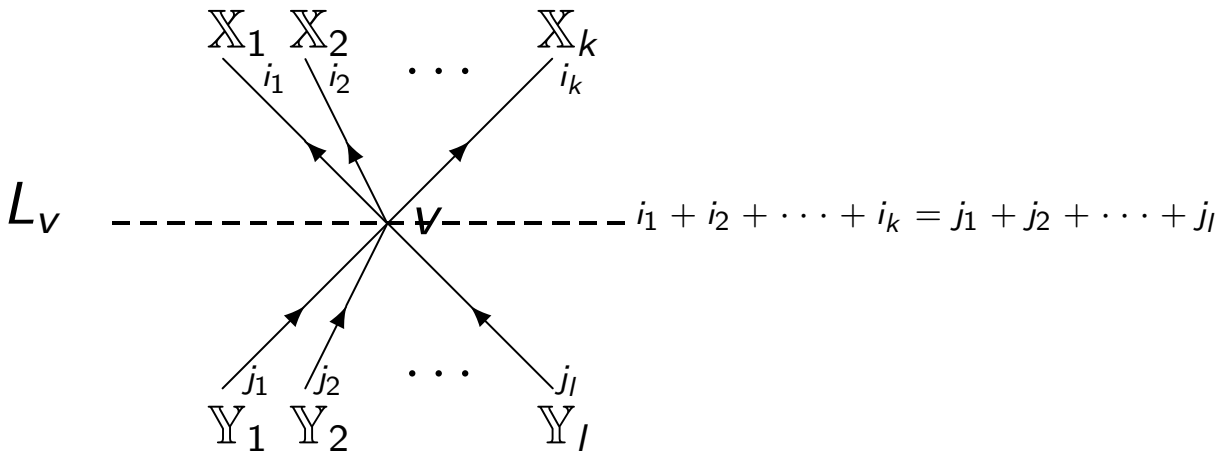


Define $R = \mathrm{Sym}(\mathbb{X}_1|\ldots|\mathbb{X}_k|\mathbb{Y}_1|\ldots|\mathbb{Y}_l)$. Write $\mathbb{X} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_k$, $\mathbb{Y} = \mathbb{Y}_1 \cup \cdots \cup \mathbb{Y}_l$. Denote by X_j and Y_j the j-th elementary symmetric polynomial in $\mathbb{X}$ and $\mathbb{Y}$.

Matrix factorization associated to a vertex



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$$C_N(v) = \begin{pmatrix} U_1 & X_1 - Y_1 \\ U_2 & X_2 - Y_2 \\ \cdots & \cdots \\ U_m & X_m - Y_m \end{pmatrix}_R \{q^{-\sum_{1\le s<t\le k} i_s i_t}\},$$

where U_j is homogeneous of degree $2N + 2 - 2j$ and the potential is $\sum_{j=1}^m (X_j - Y_j)U_j = p_{N+1}(\mathbb{X}) - p_{N+1}(\mathbb{Y})$.

Matrix factorization associated to a MOY graph

$$C_N(\Gamma) := \bigotimes_v C_N(v),$$

where v runs through all the interior vertices of Γ (including those additional 2-valent vertices.)

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More precisely, for two sub-MOY graphs Γ_1 and Γ_2 of Γ intersecting only at (some of) their open end points, let $\mathbb{W}_1, \ldots, \mathbb{W}_n$ be the alphabets associated to these common end points. Then, in the above tensor product, $C_N(\Gamma_1) \otimes C_N(\Gamma_2)$ is the tensor product $C_N(\Gamma_1) \otimes_{\mathrm{Sym}(\mathbb{W}_1|\ldots|\mathbb{W}_n)} C_N(\Gamma_2)$.

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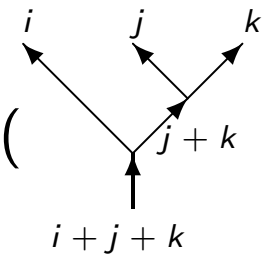
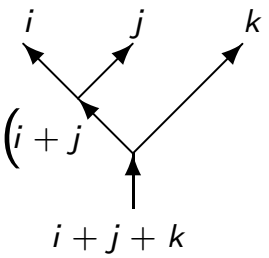
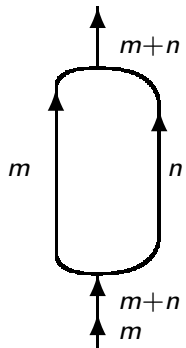
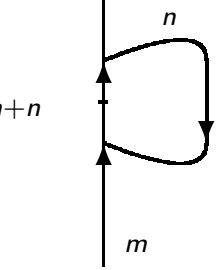
$C_N(\Gamma)$ has a $\mathbb{Z}_2$-grading and a quantum grading.

Homological MOY relations (1–4)

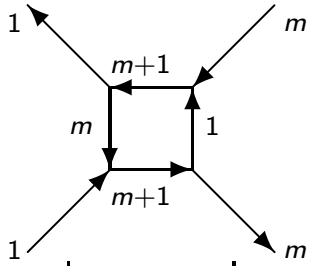
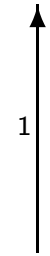
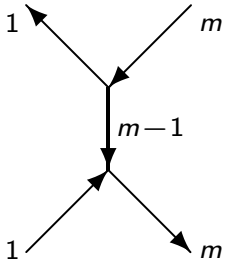
1. $C_N(\bigcirc_m) \simeq \mathbb{C}\{\begin{bmatrix} N \\ m \end{bmatrix}\}$, where $\bigcirc_m$ is a circle colored by m.

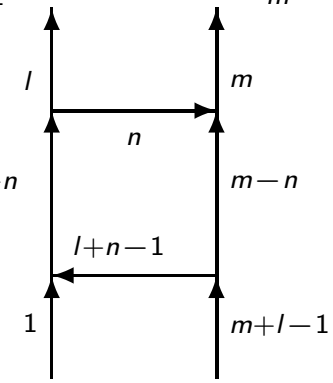
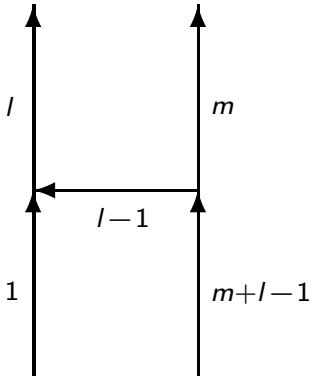
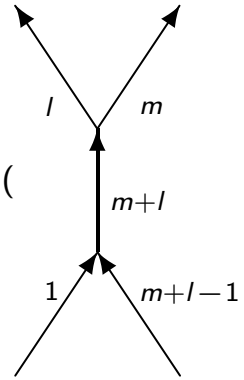
2. $C_N($  $) \simeq C_N($  $)$.

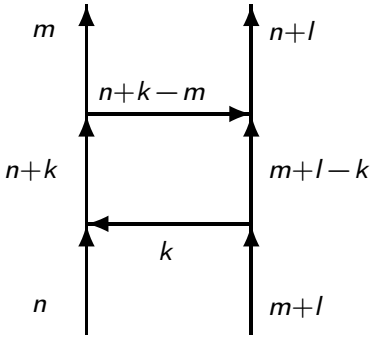
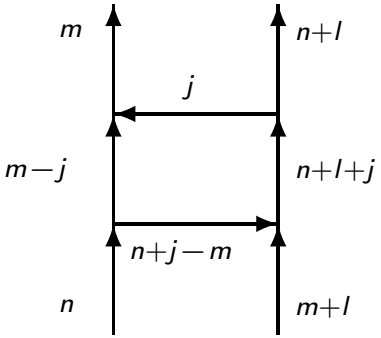
3. $C_N($  $) \simeq C_N($  $)\{\begin{bmatrix} m+n \\ n \end{bmatrix}\}$.

4. $C_N($  $) \simeq C_N($  $)\{\begin{bmatrix} N-m \\ n \end{bmatrix}\}$.

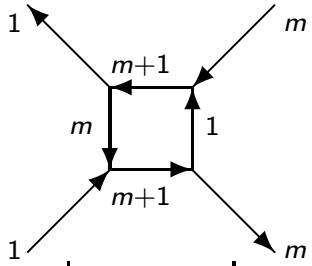
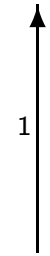
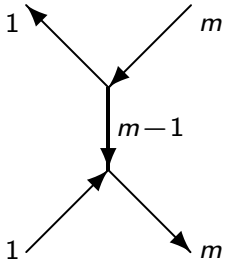
Homological MOY relations (5–7)

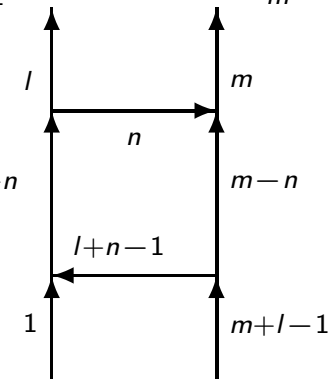
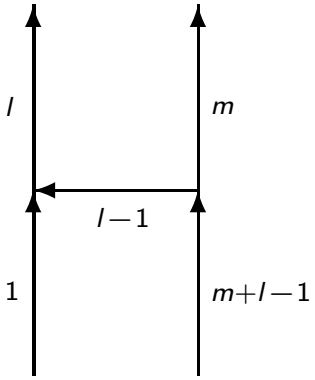
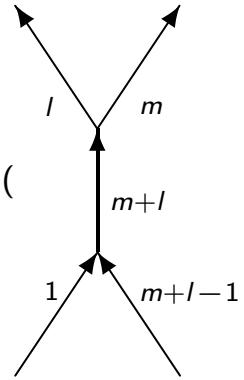
5. $C_N($  $) \simeq C_N($  $) + C_N($  $)\{[N-m-1]\}.$

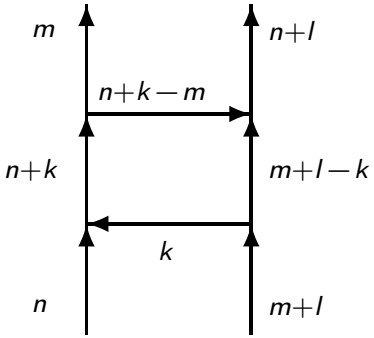
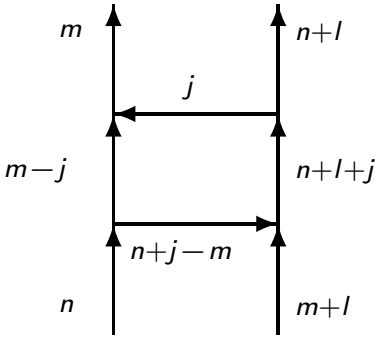
6. $C_N($  $) \simeq C_N($  $)\{\begin{bmatrix} m-1 \\ n \end{bmatrix}\} + C_N($  $)\{\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}\}.$

7. $C_N($  $) \simeq \bigoplus_{j=\max\{m-n,0\}}^{m} C_N($  $)\{\begin{bmatrix} l \\ k-j \end{bmatrix}\}.$

Homological MOY relations (5–7)

5. $C_N($  $) \simeq C_N($  $) + C_N($  $)\{[N-m-1]\}.$

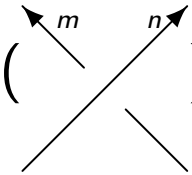
6. $C_N($  $) \simeq C_N($  $)\{\begin{bmatrix} m-1 \\ n \end{bmatrix}\} + C_N($  $)\{\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}\}.$

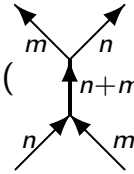
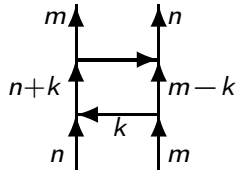
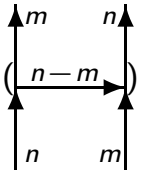
7. $C_N($  $) \simeq \bigoplus_{j=\max\{m-n,0\}}^{m} C_N($  $)\{\begin{bmatrix} l \\ k-j \end{bmatrix}\}.$

The above imply that the graded dimension of $C_N(\Gamma)$ is the $\mathfrak{sl}(N)$ MOY graph polynomial of Γ.

The chain complex of a colored crossing

Assume $n \geq m$ and temporarily forget the quantum grading shifts.

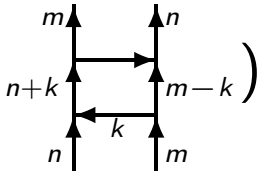
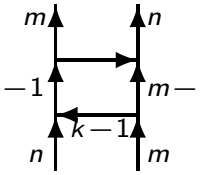
$C_N($  $)$ should be a chain complex of the form

$0 \to C_N($  $) \xrightarrow{d_m^+} \cdots \xrightarrow{d_{k+1}^+} C_N($  $) \xrightarrow{d_k^+} \cdots \xrightarrow{d_1^+} C_N($  $) \to 0,$

where d_k^+ is homogeneous of quantum degree 1.

The chain complex of a colored crossing (cont'd)

The lowest quantum grading of

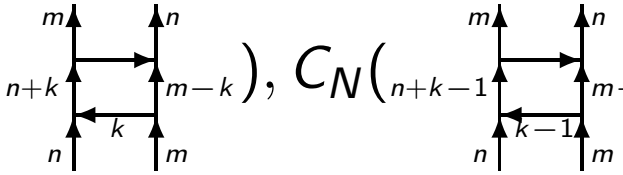
$\mathrm{Hom}_{\mathrm{HMF}}(C_N($  $), C_N($  $))$

is 1 and the space of homogeneous elements of quantum degree 1 is 1-dimensional. So d_k^+ exists and is unique up to homotopy and scaling.

The chain complex of a colored crossing (cont'd)

The lowest quantum grading of

$$\mathrm{Hom}_{\mathrm{HMF}}(C_N(\ \ ), C_N(\ \ ))$$



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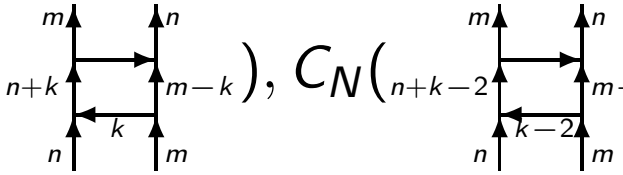
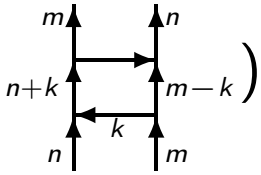
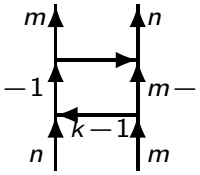


is 4. So $d_{k-1}^+ \circ d_k^+ \simeq 0$.

The chain complex of a colored crossing (cont'd)

The lowest quantum grading of

$\mathrm{Hom}_{\mathrm{HMF}}(C_N($  $), C_N($  $))$

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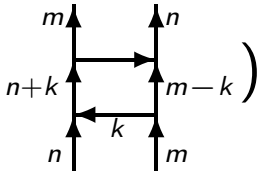
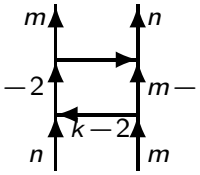
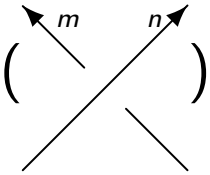
The lowest quantum grading of

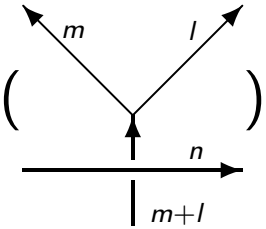
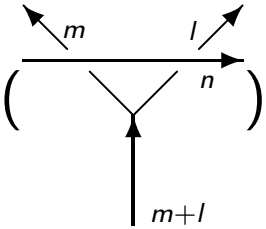
$\mathrm{Hom}_{\mathrm{HMF}}(C_N($  $), C_N($  $))$

is 4. So $d_{k-1}^+ \circ d_k^+ \simeq 0$.

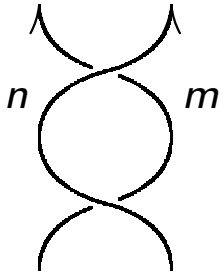
Thus, the chain complex $C_N($  $)$ exists and is unique up to chain isomorphism (if we require $d_k^+ \not\simeq 0$.)

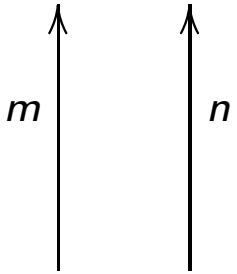
Invariance: fork sliding

Lemma

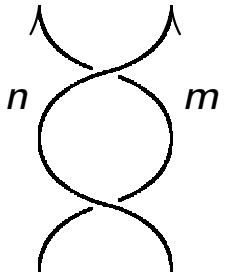
$C_N($  $) \simeq C_N($  $).$

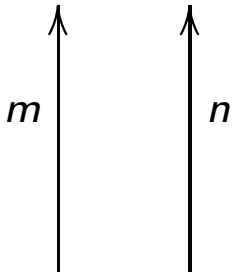
Invariance: Reidemeister moves

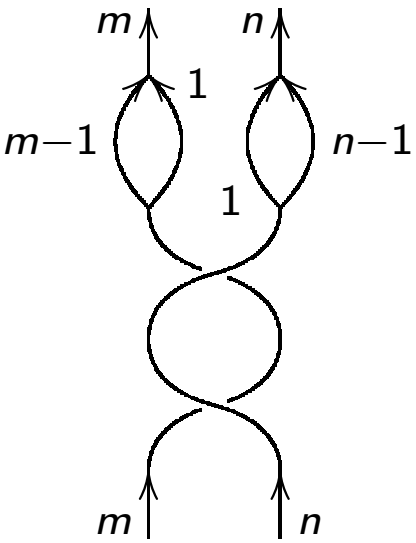


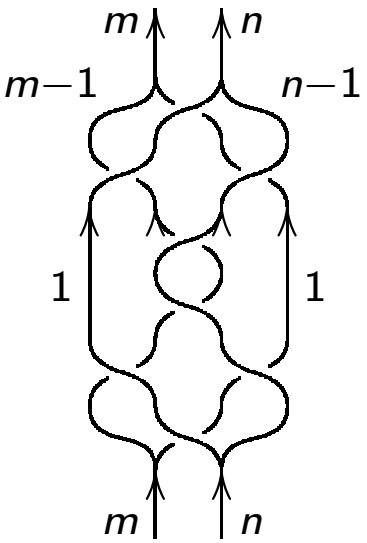


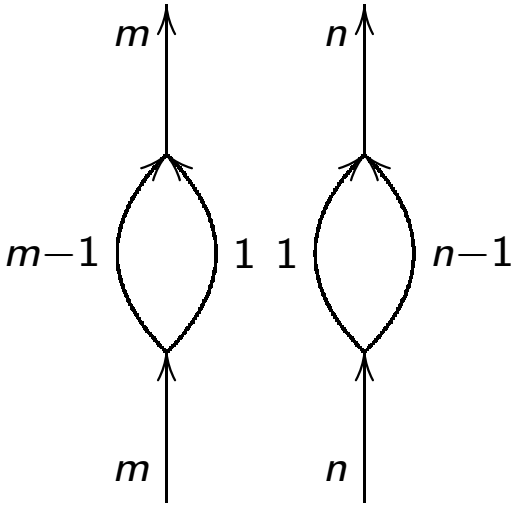
Invariance: Reidemeister moves



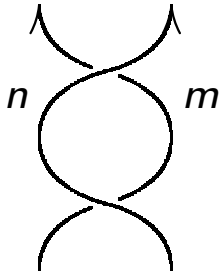


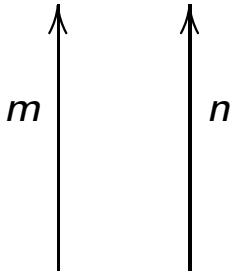


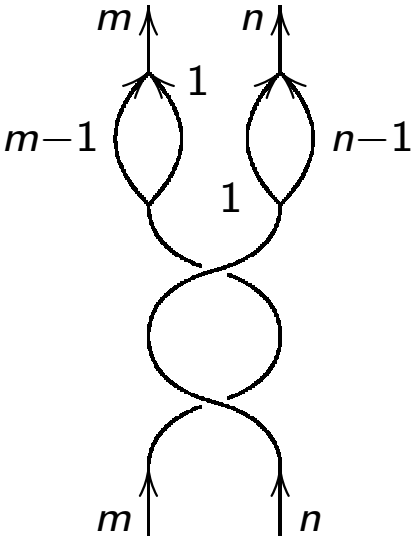


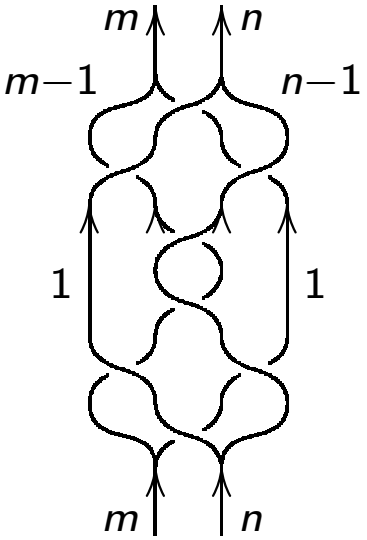


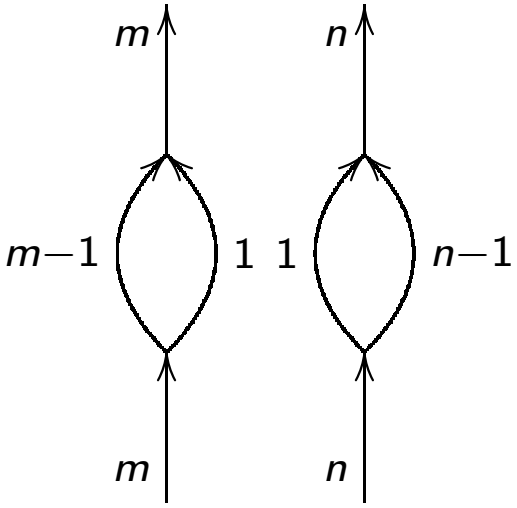
Invariance: Reidemeister moves











The invariance under Reidemeister moves II_b and III follows similarly. Reidemeister move I requires an extra lemma.

Equivariant homology

Consider the polynomial

$$f(X) = X^{N+1} + \sum_{k=1}^{N} (-1)^k \frac{N+1}{N+1-k} B_k X^{N+1-k},$$

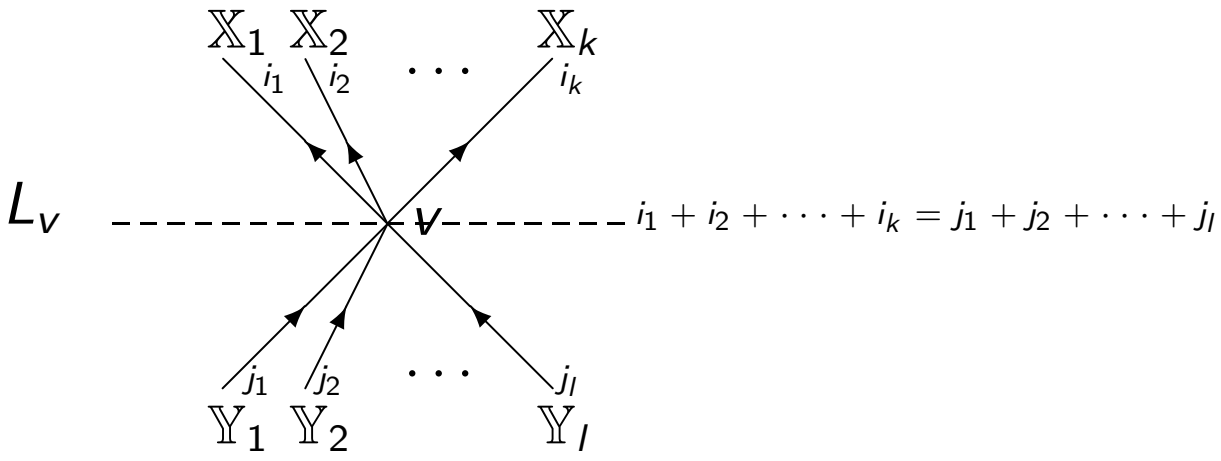
where B_k is a homogeneous indeterminate of degree $2k$.

For an alphabet $\mathbb{X} = \{x_1, \ldots, x_m\}$, define

$$f(\mathbb{X}) = \sum_{i=1}^{m} f(x_i) = p_{N+1}(\mathbb{X}) + \sum_{k=1}^{N} (-1)^k \frac{N+1}{N+1-k} B_k p_{N+1-k}(\mathbb{X}).$$

We can repeat the above construction using $f(\mathbb{X})$ instead of $p_{N+1}(\mathbb{X})$ and get an equivariant colored $\mathfrak{sl}(N)$ link homology.

Equivariant homology (cont'd)



$$C_f(v) = \begin{pmatrix} U_1 & X_1 - Y_1 \\ U_2 & X_2 - Y_2 \\ \cdots & \cdots \\ U_m & X_m - Y_m \end{pmatrix}_{R[B_1,\ldots,B_N]} \qquad \{q^{-\sum_{1\le s<t\le k} i_s i_t}\},$$

where U_j is homogeneous of degree $2N + 2 - 2j$ and the potential is $\sum_{j=1}^{m}(X_j - Y_j)U_j = f(\mathbb{X}) - f(\mathbb{Y})$.

Equivariant homology (cont'd)

The quotient map $\pi_0 : \mathbb{C}[B_1, \ldots, B_N] \to \mathbb{C}$ given by $\pi_0(B_k) = 0$ induces a functor

$$\mathrm{hmf}_{\mathbb{C}[B_1,\ldots,B_N]\otimes \mathrm{Sym}(\mathbb{X}|\mathbb{Y}), f(\mathbb{X})-f(\mathbb{Y})} \xrightarrow{\varpi_0} \mathrm{hmf}_{\mathrm{Sym}(\mathbb{X}|\mathbb{Y}), p_{N+1}(\mathbb{X})-p_{N+1}(\mathbb{Y})}.$$

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Krasner made the observation that, for any morphism ψ in $\mathrm{hmf}_{\mathbb{C}[B_1,\ldots,B_N]\otimes \mathrm{Sym}(\mathbb{X}|\mathbb{Y}), f(\mathbb{X})-f(\mathbb{Y})}$, ψ is a homotopy equivalence if and only if $\varpi_0(\psi)$ is a homotopy equivalence.

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This observation allows one to easily prove the invariance of the equivariant colored $\mathfrak{sl}(N)$ link homology using the proof of the invariance of the colored $\mathfrak{sl}(N)$ link homology.

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The quotient map $\pi : \mathbb{C}[B_1, \ldots, B_N] \to \mathbb{C}$ given by $\pi(B_k) = b_k \in \mathbb{C}$ induces a functor

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This quantum filtration induces a spectral sequence converging to H_P with E_2-page isomorphic to the (undeformed) colored $\mathfrak{sl}(N)$ link homology.

Generic deformed homology

We say that $P(X)$ is generic if

$$P'(X) = (N+1)(X^N + \sum_{k=1}^{N}(-1)^k b_k X^{N-k})$$

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For a generic P, denote by Σ the set of roots of P'. For a colored link L, a state of L is a function

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Theorem

$$H_P(L) = \bigoplus_{\psi \in \mathcal{S}_P(L)} \mathbb{C} \cdot v_\psi,$$

where $v_\psi \neq 0$ and the decomposition preserves the homological grading.

Colored $\mathfrak{sl}(N)$ Rasmussen invariants

Let P be generic. For a knot K, the m-colored $\mathfrak{sl}(N)$ Rasmussen invariant of K is

$$s_P^{(m)}(K) = \frac{1}{2}(\max\deg_q H_P(K^{(m)}) + \min\deg_q H_P(K^{(m)})),$$

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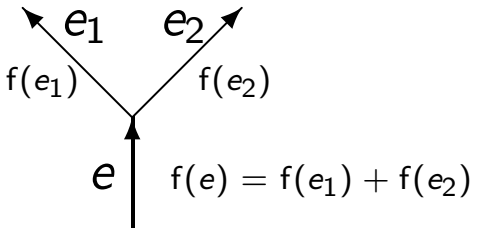
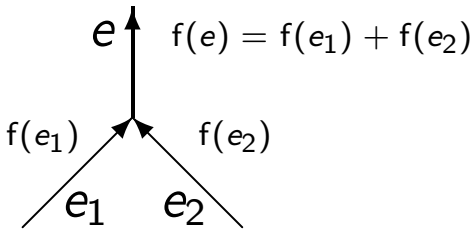
- *Quasipositive amphicheiral knots are smoothly slice.*

A composition product – labellings





Let Γ be an MOY graph. Denote by c its color function. That is, for every edge e of Γ, the color of e is $c(e)$.

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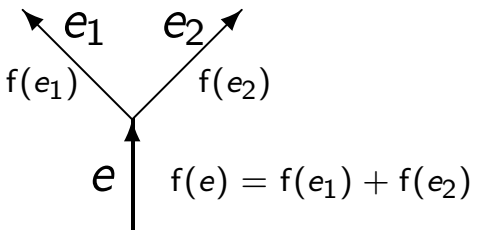

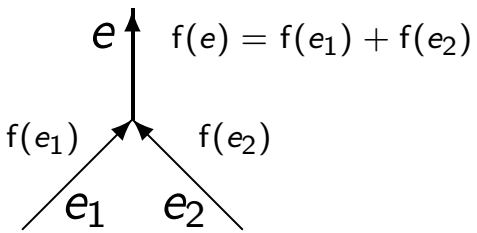

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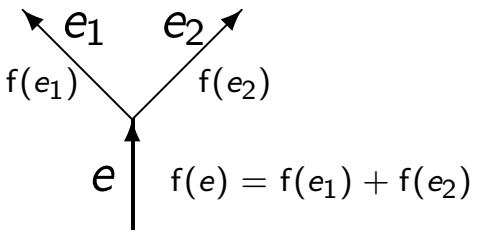

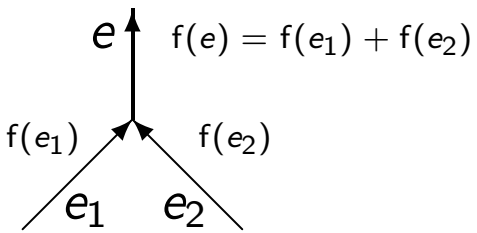

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Denote by $\mathcal{L}(\Gamma)$ the set of all labellings of Γ. For every $f \in \mathcal{L}(\Gamma)$, denote by Γ_f the MOY graph obtained by re-coloring the underlying oriented trivalent graph of Γ using f.

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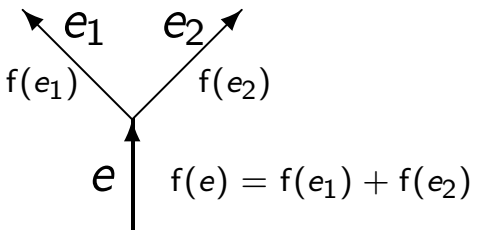

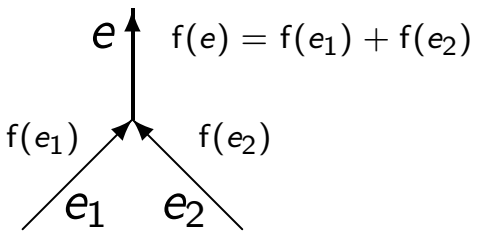

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For every $f \in \mathcal{L}(\Gamma)$, define a function $\bar{f}$ on $E(\Gamma)$ by $\bar{f}(e) = c(e) - f(e)$ for every edge e of Γ. It is easy to see that $\bar{f} \in \mathcal{L}(\Gamma)$.

A composition product

Let Γ be an MOY graph. For any positive integers M and N,

$$\langle \Gamma \rangle_{M+N} = \sum_{\mathfrak{f} \in \mathcal{L}(\Gamma)} q^{\sigma_{M,N}(\Gamma,\mathfrak{f})} \cdot \langle \Gamma_{\mathfrak{f}} \rangle_M \cdot \langle \Gamma_{\overline{\mathfrak{f}}} \rangle_N \, .$$

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This composition product is equivalent to the state sum formula of the $\mathfrak{sl}(N)$ MOY graph polynomial.

Colored homological MFW inequalities

For a closed braid B with writhe w and b strands,

$$w-b \leq \liminf_{N\to\infty} \frac{\min \deg_q H_N(B^{(m)})}{m(N-m)} \leq \limsup_{N\to\infty} \frac{\max \deg_q H_N(B^{(m)})}{m(N-m)} \leq w+b,$$

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When $m=1$, the above inequalities imply the original MFW inequality.