Transverse Khovanov-Rozansky Homologies

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A contact structure $\xi$ on an oriented 3-manifold $M$ is an oriented tangent plane distribution such that there is a 1-form $\alpha$ on $M$ satisfying $\xi = \ker \alpha$, $d\alpha|_{\xi} > 0$ and $\alpha \wedge d\alpha > 0$. Such a 1-form is called a contact form for $\xi$. 
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The standard contact structure $\xi_{st}$ on $S^3$ is given by the contact form $\alpha_{st} = dz - ydx + xdy = dz + r^2d\theta$. 
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*Every transverse link in the standard contact $S^3$ is transverse isotopic to a counterclockwise transverse closed braid around the z-axis.*

Clearly, any smooth counterclockwise closed braid around the z-axis can be smoothly isotoped into a transverse closed braid around the z-axis without changing the braid word.
The Transverse Markov Theorem

Transverse Markov moves:

- Braid group relations generated by
  - $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = \emptyset$,
  - $\sigma_i \sigma_j = \sigma_j \sigma_i$, when $|i - j| > 1$,
  - $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

- Conjugations: $\mu \leftrightarrow \eta^{-1} \mu \eta$.

- Positive stabilizations and destabilizations:
  $\mu \ (\in B_m) \leftrightarrow \mu \sigma_m \ (\in B_{m+1})$. 
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Two transverse closed braids are transverse isotopic if and only if the two braid words are related by a finite sequence of transverse Markov moves.
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Theorem (Orevkov, Shevchishin and Wrinkle)

Two transverse closed braids are transverse isotopic if and only if the two braid words are related by a finite sequence of transverse Markov moves.

So there is a one-to-one correspondence between transverse isotopy classes of transverse links and closed braids modulo transverse Markov moves.
Contact Framing

$\xi_{st}$ admits a nowhere vanishing basis $\{\partial_x + y\partial_z, \partial_y - x\partial_z\}$. For each transverse link $L$, this basis induces a contact framing of $L$. If two transverse links are transverse isotopic, then they are isotopic as framed links.
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For a transverse closed braid $B$ of a knot with writhe $w$ and $b$ strands, its contact framing is determined by its self linking number $sl(B) := w - b$. 
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If a smooth link type contains two transverse links that are isotopic as framed links but not as transverse links, then we call this smooth link type “transverse non-simple”.
Contact Framing

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An invariant for transverse links is called classical if it depends only on the framed link type of the transverse link. Otherwise, it is called non-classical or effective.
The Khovanov-Rozansky Homology

Khovanov and Rozansky introduced an approach to construct link homologies using matrix factorizations by:

1. Choose a base ring $R$ and a potential polynomial $p(x) \in R[x]$.
2. Define matrix factorizations associated to MOY graphs using this potential $p(x)$.
3. Define chain complexes of matrix factorizations associated to link diagrams using the crossing information.
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This approach has been carried out for the following potential polynomials:

- $x^{N+1} \in \mathbb{Q}[x]$ (the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology);
- $ax \in \mathbb{Q}[a, x]$ (the HOMFLYPT homology);
- $x^{N+1} + \sum_{i=1}^{N} \lambda_i x^i \in \mathbb{Q}[x]$ (deformed $\mathfrak{sl}(N)$ Khovanov-Rozansky homology);
- $x^{N+1} + \sum_{i=1}^{N} a_i x^i \in \mathbb{Q}[a_1, \ldots, a_N, x]$ (the equivariant $\mathfrak{sl}(N)$ Khovanov-Rozansky homology).
Transverse Khovanov-Rozansky Homologies

For $N \geq 1$, applying Khovanov and Rozansky’s matrix factorization construction to $ax^{N+1} \in \mathbb{Q}[a, x]$, one gets a chain complex $C_N$. For each link diagram $D$, the homology $\mathcal{H}_N(D)$ of $C_N(D)$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}^3$-graded $\mathbb{Q}[a]$-module.
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Theorem (W)

Suppose $N \geq 1$. Let $B$ be a closed braid. Every transverse Markov move on $B$ induces an isomorphism of $\mathcal{H}_N(B)$ preserving the $\mathbb{Z}_2 \oplus \mathbb{Z}^3$-graded $\mathbb{Q}[a]$-module structure.
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*Suppose $N \geq 1$. Let $B$ be a closed braid. Every transverse Markov move on $B$ induces an isomorphism of $\mathcal{H}_N(B)$ preserving the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$-graded $\mathbb{Q}[a]$-module structure.*

Therefore, by the Transverse Markov Theorem, $\mathcal{H}_N$ is an invariant for transverse links in the standard contact $S^3$.

**Question**

*Is $\mathcal{H}_N$ an effective invariant for transverse links?*
Decategorification

\[ \mathcal{P}_N(B) := \sum_{(\epsilon, i, j, k) \in \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}^3} (-1)^i \tau^\epsilon \alpha^j \xi^k \dim_{\mathbb{Q}} \mathcal{H}_{N, \epsilon, i, j, k}(B) \in \mathbb{Z}[[\alpha, \xi]][\alpha^{-1}, \xi^{-1}, \tau]/(\tau^2 - 1) \]
Decategorification

\[ \mathcal{P}_N(B) := \sum_{(\varepsilon, i, j, k) \in \mathbb{Z}_2 \oplus \mathbb{Z} \oplus 3} (-1)^i \tau^\varepsilon \alpha^j \xi^k \dim_{\mathbb{Q}} \mathcal{H}_{\varepsilon, i, j, k}^N(B) \in \mathbb{Z}[[\alpha, \xi]][\alpha^{-1}, \xi^{-1}, \tau]/(\tau^2 - 1) \]

Theorem (W)

1. \( \mathcal{P}_N \) is invariant under transverse Markov moves.

2. \( \alpha^{-1} \xi^{-N} \mathcal{P}_N(\begin{array}{c} \alpha \\ \xi \end{array}) - \alpha \xi^N \mathcal{P}_N(\begin{array}{c} \alpha \\ \xi \end{array}) = \tau(\xi^{-1} - \xi) \mathcal{P}_N(\begin{array}{c} \alpha \\ \xi \end{array}) \).

3. \( \mathcal{P}_N(U^{\uplus m}) = (\tau \alpha^{-1} [N])^m \left( \frac{1}{1 - \alpha^2} + \frac{\left( \frac{\tau \alpha \xi^{-1} + \xi^{-N}}{\xi^N - \xi^{-N}} \right)^m - 1}{\tau \alpha \xi^{-N} - \xi^{-N} + 1} \right) \), where \( U^{\uplus m} \) is the \( m \)-strand closed braid with no crossings and \([N] := \frac{\xi^{-N} - \xi^N}{\xi - 1 - \xi}\).

4. Parts 1–3 above uniquely determine the value of \( \mathcal{P}_N \) on every closed braid.
Decategorification

\[ P_N(B) := \sum_{(\epsilon, i, j, k) \in \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}^3} (-1)^i \tau^\epsilon \alpha^i \xi^k \dim_Q \mathcal{H}^{\epsilon, i, j, k}_N(B) \in \mathbb{Z}[[\alpha, \xi]][\alpha^{-1}, \xi^{-1}, \tau]/(\tau^2 - 1) \]

Theorem (W)

1. \( P_N \) is invariant under transverse Markov moves.

2. \( \alpha^{-1} \xi^{-N} P_N(\xleftarrow{\longrightarrow}) - \alpha \xi^N P_N(\xleftarrow{\longrightarrow}) = \tau(\xi^{-1} - \xi) P_N(\xleftarrow{\longrightarrow}) \).

3. \( P_N(U \sqcup^m) = (\tau \alpha^{-1} [N])^m \left( \frac{1}{1 - \alpha^2} + \frac{(\frac{\tau \alpha \xi^{-1} + \xi^{-N}}{\tau \alpha \xi^{-N} - \xi^{-N-1}})^m - 1}{\tau \alpha \xi^{-N} - \xi^{-N-1} + 1} \right) \), where \( U \sqcup^m \) is the \( m \)-strand closed braid with no crossings and \([N] := \frac{\xi^{-N} - \xi^N}{\xi^{-1} - \xi} \).

4. Parts 1–3 above uniquely determine the value of \( P_N \) on every closed braid.

It is not clear if \( P_N \) is effective. But \( P_N \) does not detect flype moves.

\( (\mu \sigma_m^k \nu \sigma_m^\pm 1 \leftrightarrow \mu \sigma_m^\pm 1 \nu \sigma_m^k, \text{ where } \mu, \nu \in B_m) \).
Module Structure

Theorem (W)

Let $H_N(B)$ be the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology of a closed braid $B$, and $(\varepsilon, i, k) \in \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}^2$.

1. $H_{N}^{\varepsilon, i, k}(B) \cong \mathcal{H}_{N}^{\varepsilon, i, *, k}(B)/(a - 1)\mathcal{H}_{N}^{\varepsilon, i, *, k}(B)$. 
Module Structure

Theorem (W)

Let \( H_N(B) \) be the \( \mathfrak{sl}(N) \) Khovanov-Rozansky homology of a closed braid \( B \), and \((\varepsilon, i, k)\) \( \in \mathbb{Z}_2 \oplus \mathbb{Z}^2 \).

1. \( H^\varepsilon,i,k_N(B) \cong H^\varepsilon,i,\ast,k_N(B)/(a-1)H^\varepsilon,i,\ast,k_N(B) \).
2. As a \( \mathbb{Z} \)-graded \( \mathbb{Q}[a] \)-module,

\[
H^\varepsilon,i,\ast,k_N(B) \cong (\mathbb{Q}[a]\{sl(B)\}_a)^{\oplus l} \oplus (\mathbb{Q}[a]\{sl(B)+2\}_a)^{\oplus (\dim \mathbb{Q} H^\varepsilon,i,k_N(B)-l)} \oplus \left( \bigoplus_{q=1}^n \mathbb{Q}[a]/(a)\{s_q\} \right),
\]

where

- \{s\}_a \text{ means shifting the a-grading by } s,
- \( l \) and \( n \) are finite non-negative integers determined by \( B \) and the triple \((\varepsilon, i, k)\),
- \( \{s_1, \ldots, s_n\} \subset \mathbb{Z} \text{ is a sequence determined up to permutation by } B \) and the triple \((\varepsilon, i, k)\),
- \( \text{sl}(B) \leq s_q \leq c_+ - c_- - 1 \text{ and } (N-1)s_q \leq k - 2N + 2c_- \text{ for } 1 \leq q \leq n \), where \( c_\pm \text{ is the number of } \pm \text{ crossings in } B \).
Negative Stabilization

Theorem (W)

Let $L$ be a transverse closed braid, and $L_-$ a transverse closed braid obtained from $L$ by a single negative stabilization. Then the chain complex $C_N(L_-)\{2,0\}$ is isomorphic to the mapping cone of the standard quotient map $\pi_0 : C_N(L) \to C_N(L)/aC_N(L)$.
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Thus, if $\mathcal{H}_N(L)$ is the homology of $C_N(L)/aC_N(L)$, there is a long exact sequence

\[ \cdots \to \mathcal{H}^{\varepsilon,i-1}_N(L)[-2,0] \xrightarrow{\pi_0} \mathcal{H}^{\varepsilon,i-1}_N(L_-) \to \mathcal{H}^{\varepsilon,i}_N(L_-) \to \mathcal{H}^{\varepsilon,i}_N(L)[-2,0] \xrightarrow{\pi_0} \cdots \]
Negative Stabilization (cont’d)

Theorem (W)
Let $B$ be a closed braid and $B_-$ a stabilization of $B$. Set $s = sl(B)$. Then for any $(i, k) \in \mathbb{Z}^{\oplus 2}$, there are a long exact sequence of $\mathbb{Z}$-graded $\mathbb{Q}[a]$-modules

$$
\cdots \rightarrow H_N^{s-1,i,*,k}(B_-) \rightarrow H_N^{s,i-1,*,k+N+1}(B)\{-1\}_a
\rightarrow H_N^{s,i-1,*,k+N+1}(B)\{-1\}_a \rightarrow 0
$$

and a short exact sequence of $\mathbb{Z}$-graded $\mathbb{Q}[a]$-modules

$$
0 \rightarrow H_N^{s,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{s\}_a \rightarrow H_N^{s,i,*,k}(B_-) \rightarrow H_N^{s-1,i-1,*,k+N+1}(B)\{-1\}_a \rightarrow 0.
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Transverse Unknots

- Bennequin’s inequality implies that the highest self linking number of a transverse unknot is \(-1\), which is attained by the 1-strand transverse closed braid.
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- Eliashberg and Fraser showed that two transverse unknots are transverse isotopic if and only if their self linking numbers are equal.
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- Denote by $U_0$ the transverse unknot with self linking $-1$ and by $U_m$ the transverse unknot obtained from $U_0$ by $m$ negative stabilizations.
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- Denote by $U_0$ the transverse unknot with self linking $-1$ and by $U_m$ the transverse unknot obtained from $U_0$ by $m$ negative stabilizations.
- Then every transverse unknot is transverse isotopic to $U_m$ for some $m \geq 0$. 
Transverse Unknots (cont’d)

\[ \mathcal{F} := \bigoplus_{l=0}^{N-1} \mathbb{Q}[a] \langle 1 \rangle \{-1, -N + 1 + 2l\}, \]

\[ \mathcal{T} := \bigoplus_{l=0}^{\infty} \mathbb{Q}[a]/(a) \langle 1 \rangle \{-1, N + 1 + 2l\}, \]
Transverse Unknots (cont’d)

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\[ T := \bigoplus_{l=0}^{\infty} \mathbb{Q}[a]/(a) \langle 1 \rangle \{ -1, N + 1 + 2l \}, \]

\[ \mathcal{H}_N(U_0) \cong F \oplus T, \]

\[ \mathcal{H}_N(U_1) \cong F \oplus T \langle 1 \rangle \{ -1, -N - 1 \| 1 \|, \]
Transverse Unknots (cont’d)

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\[ \mathcal{T} := \bigoplus_{l=0}^{\infty} \mathbb{Q}[a]/(a) \langle 1 \rangle \{-1, N + 1 + 2l\}, \]

\[ \mathcal{H}_N(U_0) \cong \mathcal{F} \oplus \mathcal{T}, \]

\[ \mathcal{H}_N(U_1) \cong \mathcal{F} \oplus \mathcal{T} \langle 1 \rangle \{-1, -N - 1\} \|

\text{and, for } m \geq 2, \]

\[ \mathcal{H}_N(U_m) \cong \mathcal{F}\{-2(m - 1), 0\} \oplus \mathcal{T} \langle m \rangle \{-m, -m(N + 1)\} \|

\bigoplus_{l=1}^{m-1} \mathcal{F}/a\mathcal{F} \langle l \rangle \{-2m + l, -l(N + 1)\} \|

\text{where } "\|l\|" \text{ means shifting the homological grading by } l. \]