Incidence Relations and Directed Cycles

Hao Wu

George Washington University
Directed graphs and directed cycles

A **directed graph** is a pair \( G = (V(G), E(G)) \) of finite sets, where

1. \( V(G) \) is the set of vertices of \( G \),
2. \( E(G) \) is the set of edges, each of which is directed.
A **directed graph** is a pair $G = (V(G), E(G))$ of finite sets, where

1. $V(G)$ is the set of vertices of $G$,
2. $E(G)$ is the set of edges, each of which is directed.

A **directed cycle** in $G$ is a closed directed path, that is, a sequence $v_0, x_0, v_1, x_1, \ldots, x_{n-1}, v_n, x_n, v_{n+1} = v_0$ satisfying

1. $v_0, v_1, \ldots, v_n$ are pairwise distinct vertices of $G$,
2. each $x_i$ is an edge of $G$ with initial vertex $v_i$ and terminal vertex $v_{i+1}$.

Two such sequences represent the same directed cycle if one is a circular permutation of the other.
Cycles packing numbers

Two directed cycles in $G$ are called edge-disjoint if they have no common edges. Two directed cycles in $G$ are called disjoint if they have no common vertices.
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For a directed graph $G$, we define

- $\alpha(G) := \text{maximal number of pairwise edge-disjoint directed cycles in } G$,
- $\tilde{\alpha}(G) := \text{maximal number of pairwise disjoint directed cycles in } G$,

$\alpha(G)$ is known as the **cycle packing number** of $G$. We call $\tilde{\alpha}(G)$ the **strong cycle packing number** of $G$. 
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$\alpha(G)$ is known as the **cycle packing number** of $G$. We call $\tilde{\alpha}(G)$ the **strong cycle packing number** of $G$.

Our goal is to determine $\alpha(G)$ and $\tilde{\alpha}(G)$ using elementary projective algebraic geometry.
Directed trials, paths and circuits

Given a directed graph \( G \), a **directed trail** in \( G \) from a vertex \( u \) to a different vertex \( v \) is a sequence

\[ u = v_0, x_0, v_1, x_1, \ldots, x_{n-1}, v_n = v \]  

such that

1. \( x_0, x_1, \ldots, x_{n-1} \) are pairwise distinct edges of \( G \),
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If, in addition, we require $v_0, v_1, \ldots, v_n$ to be pairwise distinct, then the above sequence is a **directed path**.
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A **directed circuit** in $G$ is a closed trial, that is, a sequence $v_0, x_0, v_1, x_1, \ldots, x_{n-1}, v_n, x_n, v_{n+1} = v_0$ satisfying

1. $x_0, x_1, \ldots, x_n$ are pairwise distinct edges of $G$,
2. each $x_i$ is an edge of $G$ with initial vertex $v_i$ and terminal vertex $v_{i+1}$.

Two such sequences represent the same directed circuit if one is a circular permutation of the other.
Disassembling a directed graph

Let $G$ be a directed graph, and $v$ a vertex of $G$. Assume $\deg_{\text{in}} v = n$ and $\deg_{\text{out}} v = m$. Set $k_v := \max\{m, n\}$ and $l_v := \min\{m, n\}$. 
Disassembling a directed graph

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To disassemble $G$ at $v$ is to split $v$ into $k_v$ vertices such that

1. $l_v$ of these new vertices have in-degree 1 and out degree 1.
2. $k_v - l_v$ of these new vertices have degree 1 such that
   - if $m \geq n$, then each of these degree 1 vertices has in-degree 0 and out-degree 1;
   - if $m < n$, then each of these degree 1 vertices has in-degree 1 and out-degree 0.
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To disassemble $G$ is to disassemble $G$ at all vertices of $G$. 
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To disassemble $G$ is to disassemble $G$ at all vertices of $G$.

We call each graph resulted from disassembling $G$ a **disassembly** of $G$ and denote by $\text{Dis}(G)$ the set of all disassemblies of $G$. 
Disassemblies of a directed graph

Lemma

Let $G$ be a directed graph, and $D$ a disassembly of $G$.

1. $D$ is a disjoint union of directed paths and directed cycles.
Disassemblies of a directed graph

Lemma

Let $G$ be a directed graph, and $D$ a disassembly of $G$.

1. $D$ is a disjoint union of directed paths and directed cycles.
2. $E(D) = E(G)$ and there is a natural graph homomorphism from $D$ to $G$ that maps each edge to itself and each vertex $v$ in $D$ the vertex in $G$ used to create $v$. 

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2. $E(D) = E(G)$ and there is a natural graph homomorphism from $D$ to $G$ that maps each edge to itself and each vertex $v$ in $D$ the vertex in $G$ used to create $v$.
3. Under the above natural homomorphism,
   - each directed path in $D$ is mapped to a directed trail in $G$,
   - each directed cycle in $D$ is mapped to a directed circuit in $G$,
   - the collection of all directed cycles in $D$ is mapped to a collection of pairwise edge-disjoint circuits in $G$. 
**Disassemblies of a directed graph**

**Lemma**

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3. Under the above natural homomorphism,
   - each directed path in $D$ is mapped to a directed trail in $G$,
   - each directed cycle in $D$ is mapped to a directed circuit in $G$,
   - the collection of all directed cycles in $D$ is mapped to a collection of pairwise edge-disjoint directed cycles in $G$.
4. $\alpha(D) \leq \alpha(G)$ and $\alpha(D) = \alpha(G)$ if and only if the collection of all directed cycles in $D$ is mapped to a collection of $\alpha(G)$ pairwise edge-disjoint directed cycles in $G$.  

Incidence relations, special case

Incidence relations:

\[ y \rightarrow x \Rightarrow y = x, \]
\[ x \rightarrow \Rightarrow 0 = x, \]
\[ y \rightarrow \Rightarrow y = 0. \]
Incidence relations, special case

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Let \( G \) be a directed graph, and \( D \) a disassembly of \( G \). Recall that \( E(D) = E(G) \). Define the **incidence set** of \( D \) by

\[ P(D) = \{ p \in \mathbb{P}^{E(G)^{-1}} \mid p \text{ satisfies all incidence relations in } D. \} \]
Incidence relations, special case

**Incidence relations:**

- $y \Rightarrow y = x,$
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- $y \Rightarrow y = 0.$

Let $G$ be a directed graph, and $D$ a disassembly of $G$. Recall that $E(D) = E(G)$. Define the **incidence set** of $D$ by

$$P(D) = \{ p \in \mathbb{C}P_{|E(G)|-1} | \text{ } p \text{ satisfies all incidence relations in } D. \}$$

Clearly, $P(D)$ is a linear subspace of $\mathbb{C}P_{|E(G)|-1}$. 
Lemma

Let $G$ be a directed graph.

1. For any disassembly $D$ of $G$, the incidence set $P(D)$ of $D$ is a linear subspace of dimension $\alpha(D) - 1$ of $\mathbb{CP}^{|E(G)|-1}$. 
Incidence sets of disassemblies

Lemma

Let $G$ be a directed graph.

1. For any disassembly $D$ of $G$, the incidence set $P(D)$ of $D$ is a linear subspace of dimension $\alpha(D) - 1$ of $\mathbb{C}P^{|E(G)|-1}$.

2. For any two disassemblies $D_1$ and $D_2$ of $G$, $P(D_1) = P(D_2)$ as linear subspaces of $\mathbb{C}P^{|E(G)|-1}$ if and only if, under the natural homomorphisms from $D_1$ and $D_2$ to $G$, the collections of all directed cycles in $D_1$ and $D_2$ are mapped to the same collection of pairwise edge-disjoint circuits in $G$. 
Lemma

Let $G$ be a directed graph.

1. For any disassembly $D$ of $G$, the incidence set $P(D)$ of $D$ is a linear subspace of dimension $\alpha(D) - 1$ of $\mathbb{C}P^{\left|E(G)\right|-1}$.

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The set of incidence relations at $v$ is

$$\Delta_v := \{ e_l(x_1, \ldots, x_m) = e_l(y_1, \ldots, y_n) \mid 1 \leq l \leq \max\{n, m\}, \}$$

where $e_l$ is the degree-$l$ elementary symmetric polynomial.
Incidence relations, general case

The set of *incidence relations* at \( v \) is

\[
\Delta_v := \{e_l(x_1, \ldots, x_m) = e_l(y_1, \ldots, y_n) \mid 1 \leq l \leq \max\{n, m\}\},
\]

where \( e_l \) is the degree-\( l \) elementary symmetric polynomial.

For a directed graph \( G \), its set of incidence relations is

\[
\Delta(G) := \bigcup_{v \in V(G)} \Delta_v.
\]

The *incidence set* of \( G \) is

\[
P(G) = \{p \in \mathbb{CP}^{|E(G)|-1} \mid p \text{ satisfies all incidence relations in } G\}.
\]
The incidence set

Proposition

As subsets of $\mathbb{CP}^{E(G)-1}$, $P(G) = \bigcup_{D \in \text{Dis}(G)} P(D)$. 
The incidence set

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As subsets of $\mathbb{C}P^{|E(G)|-1}$, $P(G) = \bigcup_{D \in \text{Dis}(G)} P(D)$.

Lemma
Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be two sequences of complex numbers. Then the following statements are equivalent.

1. $e_k(x_1, \ldots, x_n) = e_k(y_1, \ldots, y_n)$ for $k = 1, \ldots, n$, where $e_k$ is the $k$-th elementary symmetric polynomial.

2. There is a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $x_i = y_{\sigma(i)}$ for $i = 1, \ldots, n$. 
Proposition

Let $G$ be a directed graph.

1. For every maximal\(^1\) collection $\mathcal{C}$ of pairwise edge-disjoint directed cycles in $G$, there is a disassembly $D_\mathcal{C}$ of $G$ such that $\mathcal{C}$ is the collection of images of directed cycles in $D_\mathcal{C}$ under the natural homomorphism.

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\(^1\)with respect to the partial order of sets given by inclusion.
Irreducible components of the incidence set

Proposition

Let $G$ be a directed graph.

1. For every maximal\(^1\) collection $C$ of pairwise edge-disjoint directed cycles in $G$, there is a disassembly $D_C$ of $G$ such that $C$ is the collection of images of directed cycles in $D_C$ under the natural homomorphism.

2. For any disassembly $D$ of $G$, $P(D)$ is not a proper subset of $P(D')$ for any $D' \in \text{Dis}(G)$ if and only if the natural homomorphism maps the directed cycles in $D$ to a maximal collection of pairwise edge-disjoint directed cycles in $G$.

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Proposition

Let $G$ be a directed graph.

1. For every maximal\(^1\) collection $C$ of pairwise edge-disjoint directed cycles in $G$, there is a disassembly $D_C$ of $G$ such that $C$ is the collection of images of directed cycles in $D_C$ under the natural homomorphism.

2. For any disassembly $D$ of $G$, $P(D)$ is not a proper subset of $P(D')$ for any $D' \in \text{Dis}(G)$ if and only if the natural homomorphism maps the directed cycles in $D$ to a maximal collection of pairwise edge-disjoint directed cycles in $G$.

3. The set of irreducible components of $P(G)$ is $\{P(D_C) \mid C$ is a maximal collection of pairwise edge-disjoint directed cycles in $G\}$.

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\(^1\)with respect to the partial order of sets given by inclusion.
The incidence set determines the cycle packing number

Theorem

Let $G$ be any directed graph. Then:

1. $\dim P(G) = \alpha(G) - 1$;
The incidence set determines the cycle packing number

Theorem

Let $G$ be any directed graph. Then:

1. $\dim P(G) = \alpha(G) - 1$;

2. $\deg P(G) =$ the number of distinct collections of $\alpha(G)$ edge-disjoint cycles in $G$;
The incidence set determines the cycle packing number

Theorem
Let $G$ be any directed graph. Then:

1. $\dim P(G) = \alpha(G) - 1$;

2. $\deg P(G) =$ the number of distinct collections of $\alpha(G)$ edge-disjoint cycles in $G$;

3. There is a bijection between the set of irreducible components of $P(G)$ of dimension $n - 1$ and the set of maximal collections of pairwise edge-disjoint directed cycles in $G$ containing exactly $n$ directed cycles.
Collections of pairwise disjoint directed cycles, a stretch
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For a directed graph $G$, denote by $B_G$ obtained by stretching each vertex in $G$. 
Collections of pairwise disjoint directed cycles, a stretch

For a directed graph $G$, denote by $B_G$ obtained by stretching each vertex in $G$.

**Lemma**

1. There is a bijection from the set of directed cycles in $G$ to the set of directed cycles in $B_G$;
For a directed graph $G$, denote by $B_G$ obtained by stretching each vertex in $G$.

**Lemma**

1. *There is a bijection from the set of directed cycles in $G$ to the set of directed cycles in $B_G$;*

2. *A collection of directed cycles in $G$ is pairwise disjoint if and only if the corresponding collection in $B_G$ is pairwise edge-disjoint;*
For a directed graph $G$, denote by $B_G$ obtained by stretching each vertex in $G$.

**Lemma**

1. There is a bijection from the set of directed cycles in $G$ to the set of directed cycles in $B_G$;
2. A collection of directed cycles in $G$ is pairwise disjoint if and only if the corresponding collection in $B_G$ is pairwise edge-disjoint;
3. $\tilde{\alpha}(G) = \alpha(B_G)$. 
The strong incidence set

The set of strong incidence relations at $v$ is

$$\tilde{\Delta}_v := \{ e_1(x_1, \ldots, x_m) = e_1(y_1, \ldots, y_n) \} \cup \{ e_l(x_1, \ldots, x_m) = 0 \mid 2 \leq l \leq m \} \cup \{ e_l(y_1, \ldots, y_n) = 0 \mid 2 \leq l \leq n \}.$$
The strong incidence set

The set of **strong incidence relations** at $v$ is

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For a directed graph $G$, its set of **strong incidence relations** is $\tilde{\Delta}(G) := \bigcup_{v \in V(G)} \tilde{\Delta}_v$. The **strong incidence set** of $G$ is

$$\tilde{P}(G) = \{p \in \mathbb{C}^{|E(G)|-1} \mid p \text{ satisfies all strong incidence relations in } G\}.$$
The strong cycle packing number

Theorem

Let $G$ be any directed graph. Then:

1. $\tilde{P}(G)$ is the union of finitely many linear subspaces of $\mathbb{CP}^{|E(G)|-1}$.
The strong cycle packing number

**Theorem**

Let $G$ be any directed graph. Then:

1. $\tilde{P}(G)$ is the union of finitely many linear subspaces of $\mathbb{CP}^{|E(G)|-1}$;

2. $\dim \tilde{P}(G) = \tilde{\alpha}(G) - 1$;
The strong cycle packing number

Theorem

Let $G$ be any directed graph. Then:

1. $\tilde{P}(G)$ is the union of finitely many linear subspaces of $\mathbb{CP}^{\mid E(G)\mid - 1}$;
2. $\dim \tilde{P}(G) = \tilde{\alpha}(G) - 1$;
3. $\deg \tilde{P}(G) =$ the number of distinct collections of $\tilde{\alpha}(G)$ disjoint cycles in $G$;
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Theorem
Let $G$ be any directed graph. Then:

1. $\tilde{P}(G)$ is the union of finitely many linear subspaces of $\mathbb{C}^{\left|E(G)\right|-1}$;

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3. $\deg \tilde{P}(G) = \text{the number of distinct collections of } \tilde{\alpha}(G) \text{ disjoint cycles in } G$;

4. There is a bijection between the set of irreducible components of $\tilde{P}(G)$ of dimension $n - 1$ and the set of maximal collections of pairwise disjoint directed cycles in $G$ containing exactly $n$ directed cycles.
Irreducible incidence sets
Irreducible incidence sets

Theorem
Let $G$ be a directed graph.

1. The following statements are equivalent:
   1.1 $P(G)$ is irreducible;
   1.2 $P(G)$ is a linear subspace of $\mathbb{CP}^{|E(G)|-1}$;
   1.3 $G$ contains exactly $\alpha(G)$ distinct directed cycles.
Irreducible incidence sets

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2. The following statements are equivalent:
   2.1 $\tilde{P}(G)$ is irreducible;
   2.2 $\tilde{P}(G)$ is a linear subspace of $\mathbb{CP}^{|E(G)|-1}$;
   2.3 $G$ contains exactly $\tilde{\alpha}(G)$ distinct directed cycles.
Irreducible incidence sets

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   2.3 $G$ contains exactly $\tilde{\alpha}(G)$ distinct directed cycles.

3. If $\tilde{P}(G)$ is irreducible, then $P(G) = \tilde{P}(G)$. 
Theorem

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3. If $\tilde{P}(G)$ is irreducible, then $P(G) = \tilde{P}(G)$.

See arXiv:1508.07337 for more related results.