SOLUTIONS TO THE FINAL EXAM
MATH 455.2 SPRING 2005

1. *(15 pts)* On a square $n \times n$ grid how many ways to get from the bottom left corner to the top right corner if the only moves allowed are up or to the right?

   Every such way contains exactly $2n$ moves, n of which are "up" and n are "to the right". Thus each way can be represented by a length $2n$ string containing n letters "U" and n letters "R" in some order. The number of all such strings is $\binom{2n}{n}$.

2. *(15 pts)* Let a_n be the number of subsets of the set $\{1, 2, \ldots, n\}$ that do not contain two consecutive numbers. For example, $a_2 = 3$ (the subsets are $\emptyset$, $\{1\}$, and $\{2\}$).
   (a) *(5 pts)* Compute a_3 and a_4 by listing the corresponding subsets.

   $a_3 = 5$ ($\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$,$\{1,3\}$) and $a_4 = 8$ ($\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$,$\{1,3\}$, $\{2,4\}$,$\{1,4\}$).

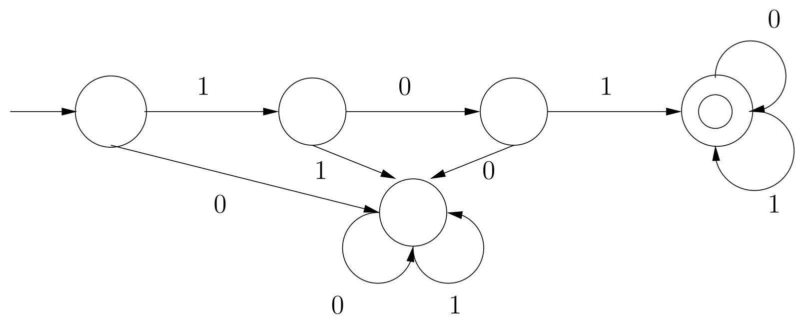
   (b) *(10 pts)* Give a recurrence relation for the numbers a_n. Explain your answer. (Hint: How many subsets do not contain the number n? How many do contain n?)

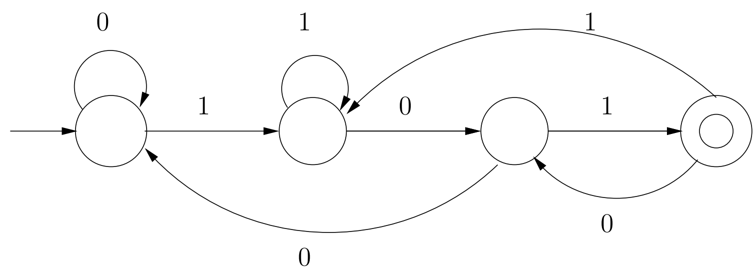
   The answer is $a_n = a_{n-1} + a_{n-2}$. Clearly, all subsets of $\{1, \ldots, n\}$ that do not contain two consecutive numbers split into two groups: those that contain n and those that don't. The number of those that don't contain n is clearly a_{n-1}. The number of those that do contain n is a_{n-2}. Indeed, each such subset cannot contain $n-1$, thus is obtained from a subset of $\{1, \ldots, n-2\}$ not containing two consecutive numbers by adding one extra element n.

3. *(15 pts)* Let A be a finite set and f, g, h function from A to itself such that f is onto and $f \circ g = f \circ h$. Is it true that $g = h$ necessarily? If yes give a proof, if no provide a counterexample.

   This is true. Indeed, any onto function $f : A \to A$ is also one-to-one (since A is finite). Therefore, for any $x \in A$ the equality $f(g(x)) = f(h(x))$ implies $g(x) = h(x)$, i.e. g is the same as h.

4. *(15 pts)* Draw a finite state automaton for the input alphabet $\{0, 1\}$ that ends up in an accepting state for those and only those strings that begin with (end in[1]) 101. Clearly indicate the initial and accepting states.





5. *(15 pts)* Define an error-correcting code with 3 bits for the message and 3 check bits by setting $x_4 = (x_1 + x_2 + 1)$ mod 2, $x_5 = (x_1 + x_3 + 1)$ mod 2, $x_6 = (x_2 + x_3 + 1)$ mod 2.

   (a) *(5 pts)* List all codewords. Is the code linear? Explain.

   $C = \{000111, 001100, 010010, 011001, 100001, 101010, 110100, 111111\}$. The code is not linear, e.g. $000111 + 001100 = 001011$ which is not a codeword.

   (b) *(5 pts)* The minimal distance between any two codewords is 3 (you do not have to check that). How many errors can the code correct? Explain.

   It can correct up to one error: $(d - 1)/2 = 1$.

   (c) *(5 pts)* Show that the code is not perfect by counting the number of words that can be corrected. Give an example of a word that cannot be corrected.

The number of words C can correct is $8 + 6 \cdot 8 = 56$. The total number of words is $2^6 = 64$. Therefore C is not perfect, e.g. 111000 cannot be corrected.

[1]There were two versions of this question

6. *(20 pts)* Let $K_{m,n}$ denote the complete bipartite graph.
   (a) *(10 pts)* For what values of m and n does the graph have an Euler circuit? Explain.

   A graph has an Euler circuit if and only if the degree of every vertex is even. Since the only degrees that appear in $K_{m,n}$ are m and n, the graph has an Euler circuit if and only if both m and n are even.

   (b) *(10 pts)* For what values of m and n does the graph have a Hamiltonian circuit? Explain.

   A Hamiltonian circuit must visit every vertex exactly once and alternate visiting vertices of each type. Thus we need $m = n$. Since there is an edge between any two vertices of different types this is the only necessary condition.

7. *(15 pts)*
   (a) *(5 pts)* Give the definition of a tree.

   A tree is a simple connected graph without non-trivial circuits.

   (b) *(10 pts)* Does there exist a tree with 10 leaves (vertices of degree 1) and 5 vertices of degree 4 (and no other vertices)? Explain.

   No. By Euler's formula the number of edges in any tree is one smaller than the number of vertices. Here the total degree is $10 + 4 \cdot 5 = 30$, hence the number of edges is 15 which is the same as the number of vertices.