DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MATH 455.2 May 17, 2005
FINAL EXAM, DURATION 2 HOURS

Your Name: _____

Your Student ID: _____

This exam paper consists of 7 questions. It has 7 pages. Show your work: all your answers must be justified. No calculators, books or notes are allowed!

1. (15) _____

2. (15) _____

3. (15) _____

4. (15) _____

5. (15) _____

6. (20) _____

7. (15) _____

TOTAL (110)

1. *(15 pts)* On a square $n \times n$ grid how many ways to get from the bottom left corner to the top right corner if the only moves allowed are up or to the right?

2. *(15 pts)* Let a_n be the number of subsets of the set $\{1, 2, \ldots, n\}$ that do not contain two consecutive numbers. For example, $a_2 = 3$ (the subsets are $\emptyset$, $\{1\}$, and $\{2\}$).

   (a) *(5 pts)* Compute a_3 and a_4 by listing the corresponding subsets.

   (b) *(10 pts)* Give a recurrence relation for the numbers a_n. Explain your answer. (Hint: How many subsets do not contain the number n? How many do contain n?)

3. *(15 pts)* Let A be a finite set and f, g, h function from A to itself such that f is onto and $f \circ g = f \circ h$. Is it true that $g = h$ necessarily? If yes give a proof, if no provide a counterexample.

4. *(15 pts)* Draw a finite state automaton for the input alphabet $\{0, 1\}$ that ends up in an accepting state for those and only those strings that begin with (end in[1]) 101. Clearly indicate the initial and accepting states.

[1]There were two versions of this question

5. *(15 pts)* Define an error-correcting code with 3 bits for the message and 3 check bits by setting $x_4 = (x_1+x_2+1) \bmod 2$, $x_5 = (x_1+x_3+1) \bmod 2$, $x_6 = (x_2+x_3+1) \bmod 2$.

   (a) *(5 pts)* List all codewords. Is the code linear? Explain.

   (b) *(5 pts)* The minimal distance between any two codewords is 3 (you do not have to check that). How many errors can the code correct? Explain.

   (c) *(5 pts)* Show that the code is not perfect by counting the number of words that can be corrected. Give an example of a word that cannot be corrected.

6. *(20 pts)* Let $K_{m,n}$ denote the complete bipartite graph.

   (a) *(10 pts)* For what values of m and n does the graph have an Euler circuit? Explain.

   (b) *(10 pts)* For what values of m and n does the graph have a Hamiltonian circuit? Explain.

7. *(15 pts)*

   (a) *(5 pts)* Give the definition of a tree.

   (b) *(10 pts)* Does there exist a tree with 10 leaves (vertices of degree 1) and 5 vertices of degree 4 (and no other vertices)? Explain.