# SOLUTIONS TO PRACTICE PROBLEMS FOR MIDTERM #2
# MATH 455.2 SPRING 2005

(1) Show that the sequence $0, 1, 3, 7, \ldots, 2^n - 1, \ldots$ satisfies the recurrence relation $c_n = 2c_{n-1} + 1$ for all $n \geq 1$.

Solution: 8.1 #11

(2) Prove that $F_n = 3F_{n-3} + 2F_{n-4}$ for all $n \geq 4$, where $F_n$ denotes the $n$-th Fibonacci number.

Solution: 8.1 #26

(3) With each step you take when climbing a staircase, you can move up either one stair or two stairs. As a result, you can climb the entire staircase taking a combination of steps that are either one- or two-stair increments. Let $c_n$ be the number of different ways to climb a staircase with $n$ stairs ($n \geq 1$). Find a recurrence relation for $c_n$. Find the general formula for $c_n$.

Solution: 8.1 #39. For the general formula compare this sequence with the Fibonacci sequence.

(4) Use iteration method to guess an explicit formula for the sequence given recursively by $a_k = \frac{a_{k-1}}{2a_{k-1}-1}$ for all $k \geq 1$ and $a_0 = 2$. Prove that your guess is correct using induction.

Solution: 8.2 #43

(5) Solve the SOLHRRCC: $r_k = 2r_{k-1} - r_{k-2}$ for $k \geq 2$ and $r_0 = 1$, $r_1 = 4$.

Solution: 8.3 #13

(6) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both 1-1, is $f + g : \mathbb{R} \to \mathbb{R}$ also 1-1? If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both onto, is $f + g : \mathbb{R} \to \mathbb{R}$ also onto? If $f : \mathbb{R} \to \mathbb{R}$ is 1-1 and $c > 0$, is $c \cdot f : \mathbb{R} \to \mathbb{R}$ also 1-1?

Solution: 7.2 #32, 33, 34

(7) How many integers must you pick to be sure that at least two of them have the same remainder when divided by 7?

Solution: 7.3 #17

(8) There are 50 students in your class. All of them are known to be between 18 and 30 years old. You want to make a bet that the class contains at least $k$ students of the same age. How large can you make $k$ and yet be sure to win your bet?

Solution: 7.3 #29

(9) On a square tray with side 1 meter you place 51 dimes. Prove that no matter how you place them there will be three dimes that can be covered with a plate of radius 15 cm.

Solution: Divide the square into 25 squares of side 20 cm each. By the PH principle there will be a square which contains 3 dimes. Cover this square with the plate (the diagonal of the square is about 20*1.4=28 cm and the diameter of the plate is 30 cm).

(10) Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is 1-1 must $f$ be 1-1? What about $g$? Prove or give a counterexample.

Solution: 7.4 #18. The function $g$ may not be 1-1. For example: $X = \{1, 2\}$, $Y = \{1, 2, 3\}$, $Z = X = \{1, 2\}$, $f(1) = 1$, $f(2) = 2$, $g(1) = 1$, $g(2) = g(3) = 2$. Then the composition $g \circ f$ is identity on $X$, but $g$ is not 1-1. Draw an arrow diagram!

(11) Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is onto must $f$ be onto? What about $g$? Prove or give a counterexample.

Solution: 7.4 #19. The function $f$ may not be onto. See the above example.

(12) Prove that any infinite set contains a countable infinite subset.

Solution: 7.5 #26.

(13) Show that the set of all finite sequences of 0's and 1's is countable.

Solution: Here is how you can enumerate all finite sequences of 0's and 1's. Let the empty sequence be number 1, length one sequences (0 and 1) be number 2 and 3, length two sequences (00, 01, 10, 11) be number 4, 5, 6, and 7, and so on (in general, length $n$ sequences will be enumerated by $2^n, 2^n+1, \ldots, 2^{n+1}-1$). Notice that this is equivalent to adding 1 in front of the sequence and taking the integer number whose binary representation is this augmented sequence.