SOLUTIONS TO MIDTERM #2
MATH 455.2 SPRING 2005

1. *(20 pts)* Consider the recurrence relation $4a_n = 4a_{n-1} - a_{n-2}$, $n \geq 2$.
   (a) *(10 pts)* Find the general form of the solution.

   The characteristic equation is $4t^2 - 4t + 1 = 0$ which has a double root $t = 1/2$. Therefore the general form is

$$a_n = L(1/2)^n + Mn(1/2)^n.$$

   (b) *(10 pts)* Give a formula for a_n valid for $n \geq 0$ that satisfies the initial conditions $a_0 = 3$, $a_1 = 2$.

   Now we solve for L and M using the initial conditions. We have $3 = a_0 = L$ and $2 = a_1 = L/2 + M/2 = 3/2 + M/2$, which gives $M = 1$. Therefore

$$a_n = 3(1/2)^n + n(1/2)^n.$$

2. *(20 pts)* Suppose $c_1, c_2, \ldots$ is a sequence with $c_0 = 1$ and, for $n \geq 1$, satisfying the recurrence relation $c_n = c_{n-1} + 3 \cdot 2^{n-1}$.
   (a) *(5 pts)* Compute c_1, c_2, c_3, and c_4.

$$c_1 = 1 + 3 \cdot 2^0 = 4, \quad c_2 = 1 + 3 \cdot 2^0 + 3 \cdot 2^1 = 10,$$
$$c_3 = 1 + 3 \cdot 2^0 + 3 \cdot 2^1 + 3 \cdot 2^2 = 22, \quad c_4 = 1 + 3 \cdot 2^0 + 3 \cdot 2^1 + 3 \cdot 2^2 + 3 \cdot 2^3 = 46.$$

   (b) *(15 pts)* Guess a formula for c_n valid for $n \geq 0$ and prove that your guess is correct using mathematical induction. Use words to explain your logic and process.

   Notice

$$c_n = 1 + 3 \cdot 2^0 + \cdots + 3 \cdot 2^{n-1} = 1 + 3(2^0 + \cdots + 2^{n-1}) = 1 + 3(2^n - 1) = 3 \cdot 2^n - 2.$$

   Now let us prove this using PMI:

   Base: If $n = 0$, $c_0 = 3 \cdot 2^0 - 2 = 1$ which coincides with the initial condition.

   Step: Assume that the formula is true for $n - 1$, i.e. $c_{n-1} = 3 \cdot 2^{n-1} - 2$. We have

$$c_n = c_{n-1} + 3 \cdot 2^{n-1} = 3 \cdot 2^{n-1} - 2 + 3 \cdot 2^{n-1} = 3 \cdot 2^n - 2,$$

   i.e. the formula is true for n. By the PMI the formula is true for any $n \geq 1$.

3. *(15 pts)* How many one-to-one functions are there from the set of 5 elements to the set of 7 elements? Justify your answer, but do not attempt to write them all down.

   The image of a one-to-one function consists of 5 distinct elements of the co-domain. Each choice of 5 elements gives rise to 5! such functions. Therefore the total number of one-to-one functions is $5!\binom{7}{5}$ (or the number of 5-permutations on the set of 7 elements $P(7,5)$).

4. *(15 pts)* Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is onto, but not one-to-one.

   For example you can take f to be $\lfloor n/2 \rfloor$, where $\lfloor x \rfloor$ is the floor function. There are many other examples. To show f is onto notice that for any integer m we have $f(2m) = \lfloor 2m/2 \rfloor = \lfloor m \rfloor = m$. But f is not one-to-one since, for example, $f(0) = f(1) = 0$.

5. *(10 pts)* Prove that $2 + 4 + 6 + \cdots + 2n$ is $\Theta(n^2)$. You may use limits.

   First, find the sum using the formula for the arithmetic series: $2+4+6+\cdots+2n = n(n+1) = n^2 + n$. Now use limits:
   $$\lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$$
   Since the limit is a finite non-zero number, $n^2 + n$ is $\Theta(n^2)$.

6. *(15 pts)* Show that in a group of 3 people whose sum of ages is 67 years there will always be two people whose sum of ages is at least 45.

   Here is one of possible solutions. Notice that there will always be a person whose age is *at most* 22 (otherwise, the sum of all ages would be at least $23 \cdot 3 = 69$, which is impossible). But then the sum of the ages of the other two people will be at least $67 - 22 = 45$.

7. *(15 pts)* Prove that a countable union of countable sets is countable, i.e. if $A_1, A_2, A_3, \dots$ are countable sets then so is $A = \cup_{i=1}^{\infty} A_i$. (Hint: Let a_{ij} be the j-th element in A_i. Recall the proof of the countability of $\mathbb{N} \times \mathbb{N}$.)

   Construct a table (infinite down and to the right) whose i-th row is the list $a_{i1}, a_{i2}, a_{i3}, \dots$ of all the elements of A_i. Then enumerate the elements in the table using the zigzag path as in the proof of the countability of $\mathbb{N} \times \mathbb{N}$. This shows that the set of all elements in the table is countable. Notice that there could be repeated elements in the table since the A_i can have common elements. The union $A = \cup_{i=1}^{\infty} A_i$ is a subset of the set of all elements in the table and hence is countable.