This exam paper consists of 7 questions. It has 6 pages. Show your work: all your answers must be justified. When proving using induction make sure you include words to describe what you are assuming and what you are trying to show.
No calculators, books or notes are allowed!

1. (20) __________
2. (20) __________
3. (15) __________
4. (15) __________
5. (10) __________
6. (15) __________
7. (15) __________

TOTAL (110)
1. (20 pts) Consider the recurrence relation $4a_n = 4a_{n-1} - a_{n-2}$, $n \geq 2$.

   (a) (10 pts) Find the general form of the solution.

   (b) (10 pts) Give a formula for $a_n$ valid for $n \geq 0$ that satisfies the initial conditions $a_0 = 3$, $a_1 = 2$. 
2. (20 pts) Suppose $c_1, c_2, \ldots$ is a sequence with $c_0 = 1$ and, for $n \geq 1$, satisfying the recurrence relation $c_n = c_{n-1} + 3 \cdot 2^{n-1}$.

(a) (5 pts) Compute $c_1$, $c_2$, $c_3$, and $c_4$.

(b) (15 pts) Guess a formula for $c_n$ valid for $n \geq 0$ and prove that your guess is correct using mathematical induction. Use words to explain your logic and process.
3. (15 pts) How many one-to-one functions are there from the set of 5 elements to the set of 7 elements? Justify your answer, but do not attempt to write them all down.

4. (15 pts) Give an example of a function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ which is onto, but not one-to-one.
5. (10 pts) Prove that $2 + 4 + 6 + \ldots + 2n$ is $\Theta(n^2)$. You may use limits.

6. (15 pts) Show that in a group of 3 people whose sum of ages is 67 years there will always be two people whose sum of ages is at least 45.
7. (15 pts) Prove that a countable union of countable sets is countable, i.e. if $A_1, A_2, A_3, \ldots$ are countable sets then so is $A = \bigcup_{i=1}^{\infty} A_i$. (Hint: Let $a_{ij}$ be the $j$-th element in $A_i$. Recall the proof of the countability of $\mathbb{N} \times \mathbb{N}$.)