1. Give a numerical value for \( \binom{10}{7} \).

\[
\binom{10}{7} = \frac{10 \cdot 9 \cdot 8}{3!} = 120.
\]

2. Give a numerical value for \( P(6,4) \).

\[
P(6,4) = 6 \cdot 5 \cdot 4 \cdot 3 = 360.
\]

3. How many ways are there to place 5 identical balls into 3 distinct boxes?

\[
\binom{5 + 3 - 1}{5} = \binom{7}{5} = \frac{7 \cdot 6}{2} = 21.
\]

4. How many ways can the 25-member cooking club choose its President, Vice President, and Secretary?

\[25 \cdot 24 \cdot 23.\]

5. How many ways can 22 players be divided into two teams of 11 for a soccer game?

\[
\binom{22}{11}
\]

(Once 11 players are picked for one team the other 11 form the other team.)

6. How many distinct arrangements of the letters in HIPPOPOTAMUS are there?

\[
\binom{12}{3} \binom{9}{2} 7! \text{ or } \frac{12!}{3! \cdot 2!}.\]

7. How many numbers less than 800,000 can be formed by rearranging the digits in 219,338?

The first digit can only be 1, 2, or 3. If it is 1 then there are 5!/2 arrangements of the other digits. If it is 2 then again 5!/2 arrangements. Finally, if it is 3 then there are 5! arrangements of the other digits. Summing up: \(2 \cdot 5!\).

Another solution: Pretend the two “3” are different (\(3_1\) and \(3_2\)). Then the answer would be \(4 \cdot 5!\) (the first digit is 1, 2, 3_1 or 3_2; the rest is a permutation of the other five digits). But now we need to recall that 3_1 is the same as 3_2, hence divide the answer by 2. We get \(2 \cdot 5!\).
(8) How many ways can you arrange the letters of the alphabet so that there are exactly 5 letters between the a and the b?

If a is placed to the left of b then there are 20 ways to place a and b together. (a can take any place from 1 to 20). There are 24! ways to place the remaining letters to the remaining 24 places. We get $20 \cdot 24!$. The same if b is placed to the left of a. Total: $2 \cdot 20 \cdot 24!$.

(9) Count the number of 5-card poker hands with 4 of the same denomination.

There are 13 ways to pick the denomination for the 4 cards, and $52 - 4 = 48$ ways to pick the 5-th card. By multiplication rule the total number is $13 \cdot 48$.

(10) What is the probability that after a pair of dice is rolled the minimum of the two numbers showing is 4?

Total number of outcomes: $6 \cdot 6 = 36$. The number of outcomes in the event: 5. (They are (4, 4), (4, 5), (4, 6), (5, 4), (6, 4).) The probability is $5/36$.

(11) How many strings of 4 letters begin or end with one of the five vowels?

The number of the strings that begin with a vowel is $5 \cdot 26^3$. The number of the strings that end with a vowel is $26^3 \cdot 5$. The number of the strings that begin and end with a vowel is $5 \cdot 26^2 \cdot 5$. By inclusion/exclusion we have $5 \cdot 26^3 + 26^3 \cdot 5 - 5 \cdot 26^2 \cdot 5$.

(12) How many integers from 1 to 300 are divisible by 4 or by 14?

There are 75 divisible by 4, 21 divisible by 14 and 10 divisible by both (i.e. divisible by 28). By inclusion/exclusion we have $75 + 21 - 10 = 86$ divisible by 4 or 14.

(13) A bag has 3 red, 5 orange, 4 green, and 7 white balls. How many distinguishable collections of 3 balls can be drawn from the bag?

There are 4 types of balls. You need to choose 3. Since there is a least 3 of each type these are 3-combinations on 4-element set with repetition allowed. The answer is

$$\binom{3 + 4 - 1}{3}.$$ 

(14) Prove by induction that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$ 

Base of induction ($n = 1$):

$$1 = 1^3 = \left(\frac{1 \cdot 2}{2}\right)^2 = 1.$$
Inductive step: Suppose it is true for $n = k$, i.e.

$$\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2.$$ 

We need to show it is true for $n = k + 1$. We have

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k + 1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k + 1)^3 = (k + 1)^2 \left(\frac{k^2}{4} + k + 1\right) = (k + 1)^2 \frac{k^2 + 4k + 4}{4} = \left(\frac{(k+1)(k+2)}{2}\right)^2.$$

By PMI the statement follows for all $n$.

(15) Prove by induction that $(1 + x)^n \geq 1 + nx$ for any positive integer $n$ and any $x \geq -1$. Base of induction ($n = 1$): $1 + x \geq 1 + x$ true for any $x$.

Inductive step: Suppose the statement is true for $n = k$:

$$(1 + x)^k \geq 1 + kx \quad \text{for any } x \geq -1.$$ 

Multiply both sides of the inequality by $1 + x$ (this won’t change the sign of the inequality since $1 + x \geq 0$). We obtain

$$(1 + x)^{k+1} \geq (1 + kx)(1 + x) = 1 + kx + x + x^2 = 1 + (k + 1)x + x^2.$$ 

Clearly $1 + (k + 1)x + x^2 \geq 1 + (k + 1)x$ since $x^2 \geq 0$. Therefore we have

$$(1 + x)^{k+1} \geq 1 + (k + 1)x,$$

i.e. the statement is true for $n = k + 1$. By PMI the statement follows for all $n$.

(16) When a group of $n$ businessmen arrives at a meeting each person shakes hands with all the other people present. Guess the number of handshakes occur. Prove your guess by induction.

Trying $n = 2, 3, 4$ one can guess that the number of handshakes is $n(n - 1)/2$. Indeed, imagine one more businessman arrives. Then he has to shake hands with all the $n$ businessmen present at the meeting. This adds $n$ more handshakes. Therefore if we assume that the formula is true for $n$ people, we get $n(n-1)/2 + n$ handshakes for $n + 1$ people. An easy computation shows that $n(n-1)/2 + n = (n + 1)n/2$, i.e. the formula is also true for $n + 1$ people.

Another argument is to count the number of greetings. Each of the $n$ people has to greet the other $n - 1$ people at the meeting, which is $n(n - 1)$ total greetings. But that’s twice as many as the number of handshakes! (When Peter and Brian shake hands Peter greets Brian and also Brian greets Peter.)