SOLUTIONS TO PRACTICE PROBLEMS FOR MIDTERM #1
MATH 455 SPRING 2004

(1) Give a numerical value for $\binom{10}{7}$.

$$\binom{10}{7} = \frac{10 \cdot 9 \cdot 8}{3!} = 120.$$

(2) Give a numerical value for $P(6,4)$.

$$P(6,4) = 6 \cdot 5 \cdot 4 \cdot 3 = 360.$$

(3) How many ways are there to place 5 identical balls into 3 distinct boxes?

$$\binom{5+3-1}{5} = \binom{7}{5} = \frac{7 \cdot 6}{2} = 21.$$

(4) How many ways can the 25-member cooking club choose its President, Vice President, and Secretary?

$$25 \cdot 24 \cdot 23.$$

(5) How many ways can 22 players be divided into two teams of 11 for a soccer game?

$$\binom{22}{11}$$

(Once 11 players are picked for one team the other 11 form the other team.)

(6) How many distinct arrangements of the letters in HIPPOPOTAMUS are there?

$$\binom{12}{3}\binom{9}{2}7! \quad \text{or} \quad \frac{12!}{3! \cdot 2!}.$$

(7) How many numbers less than 800,000 can be formed by rearranging the digits in 219,338?

The first digit can only be 1, 2, or 3. If it is 1 then there are $5!/2$ arrangements of the other digits. If it is 2 then again $5!/2$ arrangements. Finally, if it is 3 then there are $5!$ arrangements of the other digits. Summing up: $2 \cdot 5!$.

Another solution: Pretend the two "3" are different (3_1 and 3_2). Then the answer would be $4 \cdot 5!$ (the first digit is 1, 2, 3_1 or 3_2; the rest is a permutation of the other five digits). But now we need to recall that 3_1 is the same as 3_2, hence divide the answer by 2. We get $2 \cdot 5!$.

(8) How many ways can you arrange the letters of the alphabet so that there are exactly 5 letters between the a and the b?

If a is placed to the left of b then there are 20 ways to place a and b together. (a can take any place from 1 to 20). There are 24! ways to place the remaining letters to the remaining 24 places. We get $20 \cdot 24!$. The same if b is placed to the left of a. Total: $2 \cdot 20 \cdot 24!$.

(9) Count the number of 5-card poker hands with 4 of the same denomination.

There are 13 ways to pick the denomination for the 4 cards, and $52 - 4 = 48$ ways to pick the 5-th card. By multiplication rule the total number is $13 \cdot 48$.

(10) What is the probability that after a pair of dice is rolled the minimum of the two numbers showing is 4?

Total number of outcomes: $6 \cdot 6 = 36$. The number of outcomes in the event: 5. (They are $(4,4), (4,5), (4,6), (5,4), (6,4)$.) The probability is $5/36$.

(11) How many strings of 4 letters begin or end with one of the five vowels?

The number of the strings that begin with a vowel is $5 \cdot 26^3$. The number of the strings that end with a vowel is $26^3 \cdot 5$. The number of the strings that begin and end with a vowel is $5 \cdot 26^2 \cdot 5$. By inclusion/exclusion we have $5 \cdot 26^3 + 26^3 \cdot 5 - 5 \cdot 26^2 \cdot 5$.

(12) How many integers from 1 to 300 are divisible by 4 or by 14?

There are 75 divisible by 4, 21 divisible by 14 and 10 divisible by both (i.e. divisible by 28). By inclusion/exclusion we have 75+21-10=86 divisible by 4 or 14.

(13) A bag has 3 red, 5 orange, 4 green, and 7 white balls. How many distinguishable collections of 3 balls can be drawn from the bag?

There are 4 types of balls. You need to choose 3. Since there is a least 3 of each type these are 3-combinations on 4-element set with repetition allowed. The answer is

$$\binom{3+4-1}{3}.$$

(14) Prove by induction that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Base of induction ($n = 1$):

$$1 = 1^3 = \left(\frac{1 \cdot 2}{2}\right)^2 = 1.$$

Inductive step: Suppose it is true for $n = k$, i.e.

$$\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2.$$

We need to show it is true for $n = k+1$. We have

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 =$$

$$(k+1)^2\left(\frac{k^2}{4} + k + 1\right) = (k+1)^2\frac{k^2+4k+4}{4} = \left(\frac{(k+1)(k+2)}{2}\right)^2.$$

By PMI the statement follows for all n.

(15) Prove by induction that $(1+x)^n \geq 1 + nx$ for any positive integer n and any $x \geq -1$. Base of induction ($n = 1$): $1 + x \geq 1 + x$ true for any x.

Inductive step: Suppose the statement is true for $n = k$:

$$(1+x)^k \geq 1 + kx \quad \text{for any} \quad x \geq -1.$$

Multiply both sides of the inequality by $1 + x$ (this won't change the sign of the inequality since $1 + x \geq 0$). We obtain

$$(1+x)^{k+1} \geq (1+kx)(1+x) = 1 + kx + x + x^2 = 1 + (k+1)x + x^2.$$

Clearly $1 + (k+1)x + x^2 \geq 1 + (k+1)x$ since $x^2 \geq 0$. Therefore we have

$$(1+x)^{k+1} \geq 1 + (k+1)x,$$

i.e. the statement is true for $n = k+1$. By PMI the statement follows for all n.

(16) When a group of n businessmen arrives at a meeting each person shakes hands with all the other people present. Guess the number of handshakes occur. Prove your guess by induction.

Trying $n = 2, 3, 4$ one can guess that the number of handshakes is $n(n-1)/2$. Indeed, imagine one more businessman arrives. Then he has to shake hands with all the n businessmen present at the meeting. This adds n more handshakes. Therefore if we assume that the formula is true for n people, we get $n(n-1)/2+n$ handshakes for $n+1$ people. An easy computation shows that $n(n-1)/2 + n = (n+1)n/2$, i.e. the formula is also true for $n+1$ people.

Another argument is to count the number of greetings. Each of the n people has to greet the other $n-1$ people at the meeting, which is $n(n-1)$ total greetings. But that's twice as many as the number of handshakes! (When Peter and Brian shake hands Peter greets Brian and also Brian greets Peter.)