SOLUTIONS TO MIDTERM #1
MATH 455 SPRING 2005

(1) (10) Give a numerical value for $\frac{\binom{11}{4}}{P(11,2)}$.

$$\frac{11\cdot 10\cdot 9\cdot 8}{(4\cdot 3\cdot 2)(11\cdot 10)}=\frac{9}{3}=3.$$

(2) (10) How many solutions to $x_1+x_2+x_3+x_4+x_5=50$ are there in *positive* integers (i.e. each $x_i\geq 1,\ i=1,2,3,4,5$)?

This is equivalent to how many ways to place 50 balls into 5 boxes so that there is at least one ball in each box. Put one ball in each box. You can distribute the remaining $50-5=45$ balls in $\binom{45+5-1}{45}$ ways. Thus there are $\binom{49}{45}$ solutions.

(3) (10) How many distinct arrangements of the letters in REVERSE start with an E and end with an R?

The number of arrangements that start with an E and end with an R is the number of all arrangements of EVERS, i.e. $\frac{5!}{2!}$ (since there are 2 letters E).

(4) (10) How many 5-card hands from the standard 52 card deck have “three of a kind”, i.e. three cards of the same denomination and two others of different denomination? (Caution: Do not include full-house hands!)

Choose one of 13 denominations for the three of a kind. Then there are $\binom{4}{3}$ choices for their suit. Now you should add the remaining two cards. There are 48 choices for the 4th card (52 minus 4 of the first three cards' denomination) and 44 choices for the 5th (48 minus 4 of the 4th card's denomination). Since the order of the last two cards does not matter we have to divide by 2. Total:

$$13\binom{4}{3}\frac{48\cdot 44}{2}.$$

Other correct answers:

$$13\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}\quad\text{or}\quad 13\binom{4}{3}\binom{48}{2}-13\binom{4}{3}12\binom{4}{2}.$$

(5) (10) Count the number of ways 12 players can be split into 3 teams of four for a bridge tournament. (Assuming there is no order of the teams.)

Choose 4 players for team 1, 4 for team 2, and 4 for team 3. That is $\binom{12}{4}\binom{8}{4}\binom{4}{4}$. Since there is no order of the teams divide by $3!$, which is the number of all permutations of the three teams. The answer is

$$\frac{1}{3!}\binom{12}{4}\binom{8}{4}.$$

(6) (10) What is the number of 4-letter words that either start or end with a vowel? (Note: "word" means a string letters, a vowel is one of the five letters $\{a, e, i, o, u\}$.)

The number of the words that begin with a vowel is $5 \cdot 26^3$. The number of the words that end with a vowel is $26^3 \cdot 5$. The number of the words that begin and end with a vowel is $5 \cdot 26^2 \cdot 5$. By inclusion/exclusion we have $5 \cdot 26^3 + 26^3 \cdot 5 - 5 \cdot 26^2 \cdot 5$.

(7) (10) What is the probability that a number between 1 and 100 (inclusive) chosen at random will be divisible by either 2, 3 or 5?

The number of integers divisible by 2 is 50, by 3 is 33, and by 5 is 20. The number of integers divisible by 2 and 3 (i.e. by 6) is 16, by 2 and 5 (i.e. by 10) is 10, and by 3 and 5 (i.e. by 15) is 6. The number of integers divisible by all of them (i.e. by 30) is 3. By inclusion/exclusion formula we have $50+33+20-16-10-6+3 = 74$ numbers divisible by at least one of 2, 3 or 5. The probability is 74/100.

(8) (20) Prove by induction that for any $n \geq 1$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Base: $n = 1$.

$$\frac{1}{1\cdot 3} = \frac{1}{2+1} = \frac{1}{3}.$$

Inductive step: assume true for $n = k$, i.e.

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}.$$

Need to prove for $n = k+1$, i.e.

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}.$$

By assumption the left hand side of the latter equality is equal to

$$\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(k+1)(2k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3},$$

as required. By the PMI the statement is true for any $n \geq 1$.

(9) A rook is a chess piece that may move any number of unoccupied squares either vertically or horizontally.

a) How many ways to place 8 indistinguishable rooks on the chess board so that no rook is attacked by another?

It is impossible to have 2 rooks on the same row without them attacking each other. Since there are 8 rooks and 8 rows each row must have exactly one rook. The rook in the 1st row can take one of 8 places. The rook in the 2nd row can take one of 7 places (it cannot be in the same column with the 1st one), etc. The answer is $8 \cdot 7 \cdot 6 \cdots 2 \cdot 1 = 8!$.

b) (*Bonus.*) How many ways to place 8 indistinguishable rooks on the chess board so that each unoccupied square is attacked by at least one rook?

Now either in every row there should be a rook or in every column there should be a rook. (Otherwise there is a row and a column where there are no rooks. But then the intersection of this row and this column is not attacked!) If in every row there is a rook that is 8^8 possible placements (8 choices for the rook on the 1st row, 8 for the rook in the 2nd etc.) Similarly there are 8^8 placements when there is a rook in every column. By inclusion/exclusion we need to subtract the number of placements when in every row and in every column there is a rook, which is 8! by part a). Therefore, the total number is $2 \cdot 8^8 - 8!$.