(1) (10) Give a numerical value for \( \frac{11\cdot 10\cdot 9\cdot 8}{(4\cdot 3\cdot 2)(11\cdot 10)} = \frac{9}{3} = 3 \).

(2) (10) How many solutions to \( x_1 + x_2 + x_3 + x_4 + x_5 = 50 \) are there in positive integers (i.e. each \( x_i \geq 1, \ i = 1, 2, 3, 4, 5 \))?

This is equivalent to how many ways to place 50 balls into 5 boxes so that there is at least one ball in each box. Put one ball in each box. You can distribute the remaining 50 \( - 5 = 45 \) balls in \( \binom{45 + 5 - 1}{45} \) ways. Thus there are \( \binom{49}{45} \) solutions.

(3) (10) How many distinct arrangements of the letters in REVERSE start with an E and end with an R?

The number of arrangements that start with an E and end with an R is the number of all arrangements of EVERS, i.e. \( \frac{5!}{2!} \) (since there are 2 letters E).

(4) (10) How many 5-card hands from the standard 52 card deck have “three of a kind”, i.e. three cards of the same denomination and two others of different denomination? (Caution: Do not include full-house hands!)

Choose one of 13 denominations for the three of a kind. Then there are \( \binom{4}{3} \) choices for their suit. Now you should add the remaining two cards. There are 48 choices for the 4th card (52 minus 4 of the first three cards’ denomination) and 44 choices for the 5th (48 minus 4 of the 4th card’s denomination). Since the order of the last two cards does not matter we have to divide by 2. Total:

\[
13 \binom{4}{3} \frac{48 \cdot 44}{2}
\]

Other correct answers:

\[
13 \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1} \quad \text{or} \quad 13 \binom{4}{3} \binom{48}{2} - 13 \binom{4}{3} 12 \binom{4}{2}.
\]

(5) (10) Count the number of ways 12 players can be split into 3 teams of four for a bridge tournament. (Assuming there is no order of the teams.)

Choose 4 players for team 1, 4 for team 2, and 4 for team 3. That is \( \binom{12}{4} \binom{8}{4} \binom{4}{4} \). Since there is no order of the teams divide by 3!, which is the number of all
permutations of the three teams. The answer is
\[
\frac{1}{3!} \binom{12}{4} \binom{8}{4}.
\]

(6) (10) What is the number of 4-letter words that either start or end with a vowel? (Note: “word” means a string letters, a vowel is one of the five letters \{a, e, i, o, u\}.)

The number of the words that begin with a vowel is \(5 \cdot 26^3\). The number of the words that end with a vowel is \(26^3 \cdot 5\). The number of the words that begin and end with a vowel is \(5 \cdot 26^2 \cdot 5\). By inclusion/exclusion we have \(5 \cdot 26^3 + 26^3 \cdot 5 - 5 \cdot 26^2 \cdot 5\).

(7) (10) What is the probability that a number between 1 and 100 (inclusive) chosen at random will be divisible by either 2, 3 or 5?

The number of integers divisible by 2 is 50, by 3 is 33, and by 5 is 20. The number of integers divisible by 2 and 3 (i.e. by 6) is 16, by 2 and 5 (i.e. by 10) is 10, and by 3 and 5 (i.e. by 15) is 6. The number of integers divisible by all of them (i.e. by 30) is 3. By inclusion/exclusion formula we have \(50 + 33 + 20 - 16 - 10 - 6 + 3 = 74\) numbers divisible by at least one of 2, 3 or 5. The probability is \(74/100\).

(8) (20) Prove by induction that for any \(n \geq 1\)
\[
\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.
\]

Base: \(n = 1\).
\[
\frac{1}{1 \cdot 3} = \frac{1}{2+1} = \frac{1}{3}.
\]

Inductive step: assume true for \(n = k\), i.e.
\[
\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}.
\]

Need to prove for \(n = k + 1\), i.e.
\[
\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2k+1)(2k+3)} = \frac{k + 1}{2k+3}.
\]

By assumption the left hand side of the latter equality is equal to
\[
\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3) + 1}{(2k+1)(2k+3)}
\]
\[
= \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)} = \frac{(k+1)(2k+1)}{(2k+1)(2k+3)} = \frac{k + 1}{2k+3},
\]
as required. By the PMI the statement is true for any \(n \geq 1\).

(9) A rook is a chess piece that may move any number of unoccupied squares either vertically or horizontally.
a) How many ways to place 8 indistinguishable rooks on the chess board so that no rook is attacked by another?

It is impossible to have 2 rooks on the same row without them attacking each other. Since there are 8 rooks and 8 rows each row must have exactly one rook. The rook in the 1st row can take one of 8 places. The rook in the 2nd row can take one of 7 places (it cannot be in the same column with the 1st one), etc. The answer is \(8\cdot 7\cdot 6\cdot \ldots \cdot 2\cdot 1 = 8!\).

b) \(\text{(Bonus.)}\) How many ways to place 8 indistinguishable rooks on the chess board so that each unoccupied square is attacked by at least one rook?

Now either in every row there should be a rook or in every column there should be a rook. (Otherwise there is a row and a column where there are no rooks. But then the intersection of this row and this column is not attacked!) If in every row there is a rook that is \(8^8\) possible placements (8 choices for the rook on the 1st row, 8 for the rook in the 2nd etc.) Similarly there are \(8^8\) placements when there is a rook in every column. By inclusion/exclusion we need to subtract the number of placements when in every row and in every column there is a rook, which is \(8!\) by part a). Therefore, the total number is \(2\cdot 8^8 - 8!\).