MATH 455 Section 2 Exam 2
April, 2006

1. X is a finite set of 5 elements, and Y is a finite set of 10 elements.
(a) (10 points) How many functions from X to Y are there?
(b) (10 points) How many injective functions from X to Y are there?
Solution. (a) $10^5 = 100,000$. (b) $10 \times 9 \times 8 \times 7 \times 6 = 30,240$.

2. (20 points) Find the sequence $\{a_n\}_{n=0}^\infty$ satisfying
$$\begin{cases} a_n = a_{n-1} + 6a_{n-2}, & n \geq 2, \\ a_0 = 2, a_1 = 1. \end{cases}$$
Solution. Solving $t^2 = t + 6$, we get $t = 3$ or $-2$. So
$$a_n = C \cdot 3^n + D \cdot (-2)^n, \quad \forall \ n \geq 0.$$ 
Specially, when $n = 0, 1$, we get
$$\begin{cases} C + D = a_0 = 2 \\ 3C - 2D = a_1 = 1 \end{cases}$$
This gives $C = D = 1$. So
$$a_n = 3^n + (-2)^n, \quad \forall \ n \geq 0.$$ 

3. (20 points) Find the sequence $\{b_n\}_{n=0}^\infty$ satisfying
$$\begin{cases} b_n = 3b_{n-1} + 2 \cdot 3^n, & n \geq 1, \\ b_0 = 0. \end{cases}$$
Solution. From the recurrence relation, we have
$$\begin{align*}
b_n &= 3b_{n-1} + 2 \cdot 3^n, \\
3b_{n-1} &= 3^2b_{n-2} + 3 \cdot 2 \cdot 3^{n-1}, \\
\cdots & \cdots \\
3^{n-k}b_k &= 3^{n-(k-1)}b_{k-1} + 3^{n-k} \cdot 2 \cdot 3^k, \\
\cdots & \cdots \\
3^{n-1}b_1 &= 3^n b_0 + 3^{n-1} \cdot 2 \cdot 3.
\end{align*}$$
Note that the constant term in each of these equations is $2 \cdot 3^n$. When we add all these equations together, the middle terms are canceled, which gives
$$b_n = 3^n b_0 + \sum_{k=1}^{n} 2 \cdot 3^k = 2n \cdot 3^n, \quad \forall \ n \geq 0.$$ 

4. (20 points) $X = \{1, 2, 3, 4, \ldots, 14, 15\}$. Find the least integer $n$ so that any subset of $X$ with $n$ or more elements contains 3 consecutive integers, and justify your answer. (Hint. you need show: (i) There is a subset of $X$ with $n - 1$ elements that does not contain 3 consecutive integers; (ii) Any subset of $X$ with $n$ or more elements contains 3 consecutive integers.)
Solution. \( n = 11 \).

(i) The set \( \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14\} \) is a subset of \( X \) with 10 elements which contains no 3 consecutive integers.

(ii) Define \( A = \{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}, \{13, 14, 15\} \}. \n
Let \( Y \) be any subset of \( X \) with 11 or more elements. Define \( f : Y \to A \) by mapping each element \( y \) of \( Y \) to the element of \( A \) containing \( y \). (E.g. \( f(1) = \{1, 2, 3\} \).)

Since \( N(Y) \geq 11 > 10 = 2 \cdot N(A) \), by the Pigeonhole Principle, there is an element of \( A \) whose pre-image under \( f \) contains at least \( 2 + 1 = 3 \) elements. This means all three numbers in this element of \( A \) are contained in \( Y \). So \( Y \) contains 3 consecutive integers.

5. (20 points) The sequence \( \{f_n\}_{n=1}^{\infty} \) satisfies

\[
\begin{align*}
  f_n &= 4f_{\left\lfloor \frac{n}{2} \right\rfloor}, \quad n \geq 2, \\
  f_1 &= 1,
\end{align*}
\]

where \( \lfloor \cdot \rfloor \) is the floor function.

Show by strong mathematical induction that \( f_n \leq n^2 \) for any positive integer \( n \).

Proof. (i) When \( n = 1 \), we have \( f_1 = 1 \leq 1^2 \). So the property is true for \( n = 1 \).

(ii) Assume that \( f_k \leq k^2 \), \( \forall k \geq 1 \), for some \( n \geq 1 \). Consider \( f_{n+1} \).

It’s easy to check that \( 1 \leq \left\lfloor \frac{n+1}{2} \right\rfloor \leq n, \forall n \geq 1 \). So the induction hypothesis applies to \( k = \left\lfloor \frac{n+1}{2} \right\rfloor \). And we have

\[
f_{n+1} = 4f_{\left\lfloor \frac{n+1}{2} \right\rfloor} \leq 4\left( \frac{n+1}{2} \right)^2 \leq 4\left( \frac{n+1}{2} \right)^2 = (n+1)^2.
\]

This shows the property is true for \( n + 1 \).

(i) and (ii) show that \( f_n \leq n^2, \forall n \geq 1 \).

6. Let \( S_0 = \phi \), and \( S_n = \{1, 2, 3, \ldots, n\} \) for \( n \geq 1 \). Define \( a_n \) to be the number of subsets of \( S_n \) that contain no consecutive integers.

(a) (8 points) Compute \( a_0, a_1, a_2 \) and \( a_3 \). (Hint. The empty set \( \phi \) does not contain consecutive integers.)

(b) (12 points) Find a recurrence relation for \( \{a_n\}_{n=0}^{\infty} \), and use it to give a general formula for \( a_n \).

Solution. (a) \( a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 5 \).

(b) For \( n \geq 2 \), divide subsets of \( S_n \) containing no consecutive integers into 2 types: (I) Subsets of \( S_n \) that do not contain \( n \) or consecutive integers; (II) Subsets of \( S_n \) that contain \( n \) but not consecutive integers.

A subset of \( S_n \) is of type (I) if and only if it is a subset of \( S_{n-1} \) containing no consecutive integers. So there are exactly \( a_{n-1} \) type (I) subsets of \( S_n \). A subset of \( S_n \) is of type (II) if and only if it is the union of \( \{n\} \) and a subset of \( S_{n-2} \) containing no consecutive integers. So there are exactly \( a_{n-2} \) type (II) subsets of \( S_{n} \). Thus, \( \{a_n\}_{n=0}^{\infty} \) satisfies the recurrence relation

\[
a_n = a_{n-1} + a_{n-2}, \quad \forall n \geq 2.
\]

Using the initial values \( a_0 = 1, a_1 = 2 \), it’s easy to see that \( a_n = F_{n+1}, \) where \( \{F_n\}_{n=0}^{\infty} \) is the Fibonacci sequence. Thus,

\[
a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2}, \quad \forall n \geq 0.
\]