MATH 455 Section 2 Exam 2

April, 2006

1. X is a finite set of 5 elements, and Y is a finite set of 10 elements.
(a) (*10 points*) How many functions from X to Y are there?
(b) (*10 points*) How many injective functions from X to Y are there?
Solution. (a) $10^5 = 100,000$. (b) $10 \times 9 \times 8 \times 7 \times 6 = 30,240$.

2. (*20 points*) Find the sequence $\{a_n\}_{n=0}^{\infty}$ satisfying

$$\begin{cases} a_n = a_{n-1} + 6a_{n-2}, \ n \geq 2, \\ a_0 = 2, \ a_1 = 1. \end{cases}$$

Solution. Solving $t^2 = t + 6$, we get $t = 3$ or -2. So

$$a_n = C \cdot 3^n + D \cdot (-2)^n, \ \forall \ n \geq 0.$$

Specially, when $n = 0, \ 1$, we get

$$\begin{cases} C + D & = & a_0 = 2 \\ 3C - 2D & = & a_1 = 1 \end{cases}$$

This gives $C = D = 1$. So

$$a_n = 3^n + (-2)^n, \ \forall \ n \geq 0.$$

3. (*20 points*) Find the sequence $\{b_n\}_{n=0}^{\infty}$ satisfying

$$\begin{cases} b_n = 3b_{n-1} + 2 \cdot 3^n, \ n \geq 1, \\ b_0 = 0. \end{cases}$$

Solution. From the recurrence relation, we have

$$\begin{array}{rcl} b_n & = & 3b_{n-1} + 2 \cdot 3^n, \\ 3b_{n-1} & = & 3^2 b_{n-2} + 3 \cdot 2 \cdot 3^{n-1}, \\ \cdots & \cdots & \cdots \\ 3^{n-k} b_k & = & 3^{n-(k-1)} b_{k-1} + 3^{n-k} \cdot 2 \cdot 3^k, \\ \cdots & \cdots & \cdots \\ 3^{n-1} b_1 & = & 3^n b_0 + 3^{n-1} \cdot 2 \cdot 3. \end{array}$$

Note that the constant term in each of these equations is $2 \cdot 3^n$. When we add all these equations together, the middle terms are canceled, which gives

$$b_n = 3^n b_0 + \sum_{k=1}^{n} 2 \cdot 3^n = 2n \cdot 3^n, \ \forall \ n \geq 0.$$

4. (*20 points*) $X = \{1, \ 2, \ 3, \ 4, \cdots, \ 14, \ 15\}$. Find the least integer n so that any subset of X with n or more elements contains 3 consecutive integers, and justify your answer. (*Hint.* you need show: `(i)` There is a subset of X with $n-1$ elements that does not contain 3 consecutive integers; `(ii)` Any subset of X with n or more elements contains 3 consecutive integers.)

Solution. $n = 11$.

(i) The set $\{1,\ 2,\ 4,\ 5,\ 7,\ 8,\ 10,\ 11,\ 13,\ 14\}$ is a subset of X with 10 elements which contains no 3 consecutive integers.

(ii) Define $A = \{\{1,\ 2,\ 3\},\ \{4,\ 5,\ 6\},\ \{7,\ 8,\ 9\},\ \{10,\ 11,\ 12\},\ \{13,\ 14,\ 15\}\}$. Let Y be any subset of X with 11 or more elements. Define $f : Y \to A$ by mapping each element y of Y to the element of A containing y. (E.g. $f(1) = \{1,\ 2,\ 3\}$.) Since $N(Y) \geq 11 > 10 = 2 \cdot N(A)$, by the Pigeonhole Principle, there is an element of A whose pre-image under f contains at least $2 + 1 = 3$ elements. This means all three numbers in this element of A are contained in Y. So Y contains 3 consecutive integers.

5. (*20 points*) The sequence $\{f_n\}_{n=1}^{\infty}$ satisfies

$$\begin{cases} f_n = 4f_{[\frac{n}{2}]},\ n \geq 2, \\ f_1 = 1, \end{cases}$$

where $[*]$ is the floor function.

Show by strong mathematical induction that $f_n \leq n^2$ for any positive integer n.

Proof. (i) When $n = 1$, we have $f_1 = 1 \leq 1^2$. So the property is true for $n = 1$.

(ii) Assume that $f_k \leq k^2$, $\forall\ k$ with $1 \leq k \leq n$, for some $n \geq 1$. Consider f_{n+1}. It's easy to check that $1 \leq [\frac{n+1}{2}] \leq n$, $\forall\ n \geq 1$. So the induction hypothesis applies to $k = [\frac{n+1}{2}]$. And we have

$$f_{n+1} = 4f_{[\frac{n+1}{2}]} \leq 4[\frac{n+1}{2}]^2 \leq 4(\frac{n+1}{2})^2 = (n+1)^2.$$

This shows the property is true for $n + 1$.

(i) and (ii) show that $f_n \leq n^2$, $\forall\ n \geq 1$.

6. Let $S_0 = \phi$, and $S_n = \{1,\ 2,\ 3, \cdots,\ n\}$ for $n \geq 1$. Define a_n to be the number of subsets of S_n that contain no consecutive integers.

(a) (*8 points*) Compute a_0, a_1, a_2 and a_3. (*Hint.* The empty set ϕ does not contain consecutive integers.)

(b) (*12 points*) Find a recurrence relation for $\{a_n\}_{n=0}^{\infty}$, and use it to give a general formula for a_n.

Solution. (a) $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_3 = 5$.

(b) For $n \geq 2$, divide subsets of S_n containing no consecutive integers into 2 types: (I) Subsets of S_n that does not contain n or consecutive integers; (II) Subsets of S_n that contains n but not consecutive integers.

A subset of S_n is of type (I) if and only if it is a subset of S_{n-1} containing no consecutive integers. So there are exactly a_{n-1} type (I) subsets of S_n. A subset of S_n is of type (II) if and only if it is the union of $\{n\}$ and a subset of S_{n-2} containing no consecutive integers. So there are exactly a_{n-2} type (II) subsets of S_n. Thus, $\{a_n\}_{n=0}^{\infty}$ satisfies the recurrence relation

$$a_n = a_{n-1} + a_{n-2},\ \forall\ n \geq 2.$$

Using the initial values $a_0 = 1$, $a_1 = 2$, it's easy to see that $a_n = F_{n+1}$, where $\{F_n\}_{n=0}^{\infty}$ is the Fibonacci sequence. Thus,

$$a_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{n+2} - (\frac{1-\sqrt{5}}{2})^{n+2}),\ \forall\ n \geq 0.$$