MATH 455 Section 2 Exam 1

March, 2006

1. (10 points) Find the value of

$$\frac{P(30,5)}{\binom{30}{5}}.$$

Solution.

$$\binom{30}{5} = \frac{\frac{30!}{(30-5)!}}{\frac{30!}{5!(30-5)!}} = 5! = 120.$$

2. (10 points) How many distinct arrangements of the letters in MASSACHUSETTS start with MA?

Solution.

$$\frac{11!}{4!2!1!1!1!1!1!} = 831600.$$

3. There are 18 teams in Italian League A Soccer. Every pair of teams play each other twice in a season.

(a) (10 points) How many league games does an Italian League A soccer team play in a season?

(b) (10 points) Altogether, how many Italian League A Soccer games are there each season?

Solution. (a) $17 \times 2 = 34$. (b) $34 \times 18/2 = 306$.

4. (10 points) X is a set of 10 elements. How many subsets of X consist of odd number of elements?

Solution.

$$\binom{10}{1} + \binom{10}{3} + \binom{10}{5} + \binom{10}{7} + \binom{10}{9} = 512.$$

5. (10 points) A straight is a poker hand consisting of 5 cards whose ranks form a sequence. The highest possible straight is AKQJ10 (Broadway) and the lowest straight is 54321 (the wheel). How many 5-card hands from the standard 52 card deck are straights (including straight flushes, royal flushes and ect.)?

Solution. The lowest rank of a straight can be $1, 2, \cdots, 10$. So there are 10 choices for the ranks of a straight. After determining the ranks, there are $4^5 = 1024$ choices for the suits of the five cards. Altogether, there are $10 \times 1024 = 10240$ straights.

6. (a) (10 points) How many solutions to $x_1 + x_2 + x_3 = 17$ are there in non-negative integers (i.e. $x_i \geq 0$ $i = 1, 2, 3$)?

(b) (10 points) How many solutions to $x_1 + x_2 + x_3 = 17$ in integers satisfy $0 \leq x_i \leq 7$, $i = 1, 2, 3$.

Solution. (a)

$$\binom{17+3-1}{3-1} = \frac{19 \times 18}{2} = 171.$$

(b) Method 1. For $i = 1, 2, 3$, let

$$A_i = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1, x_2, x_3 \geq 0,\ x_i \geq 8, x_1 + x_2 + x_3 = 17\}.$$

Then

$$\begin{aligned} N(A_i) &= \text{number of non-negative integer solutions to } y_1 + y_2 + y_3 = 9 \\ &= \binom{9+3-1}{3-1} = \binom{11}{2} = 55. \end{aligned}$$

And, when $i \neq j$,

$$\begin{aligned} N(A_i \cap A_j) &= \text{number of non-negative integer solutions to } y_1 + y_2 + y_3 = 1 \\ &= \binom{1+3-1}{3-1} = \binom{3}{2} = 3. \end{aligned}$$

And, clearly, $N(A_1 \cap A_2 \cap A_3) = 0$. So, by Inclusion/Exclusion Rule,

$$N(A_1 \cup A_2 \cup A_3) = 3 \times 55 - 3 \times 3 + 0 = 156.$$

Let

$$A = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1, x_2, x_3 \geq 0,\ x_1 + x_2 + x_3 = 17\}.$$

Then, $A_1 \cup A_2 \cup A_3 \subset A$, and

$$A \setminus (A_1 \cup A_2 \cup A_3) = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 0 \leq x_1, x_2, x_3 \leq 7,\ x_1 + x_2 + x_3 = 17\}.$$

From (a), we have $N(A) = 171$. Thus, $N(A \setminus (A_1 \cup A_2 \cup A_3)) = 171 - 156 = 15$.

Method 2. Let $M = \max\{x_1, x_2, x_3\}$. Then $M \geq \frac{17}{3} > 5$. But $M \leq 7$. So $M = 6, 7$.

If $M = 6$, then the sum of the other two numbers is 11, and both numbers ≤ 6. So the other two numbers are 6 and 5. And the multiset $[x_1, x_2, x_3] = [6, 6, 5]$, which gives 3 solutions to the equation.

If $M = 7$, then the sum of the other two numbers is 10, and both numbers ≤ 7. So the multiset $[x_1, x_2, x_3] = [7, 7, 3]$ or $[7, 6, 4]$ or $[7, 5, 5]$. Each of $[7, 7, 3]$ and $[7, 5, 5]$ gives 3 solutions to the equation, and $[7, 6, 4]$ gives 6. So there are $3 + 3 + 3 + 6 = 15$ integer solutions satisfy the conditions.

7. (10 points) Six different letters are intended for six different recipients. How many different ways are there to send these letters so that three of the recipients get right letters and the other three get wrong letters? (Every recipient must get exactly one of these six letters.)

Solution. There are $\binom{6}{3} = 20$ ways to choose the 3 recipients who get right letters. After sending the right letters, there are 2 ways to send the other 3 letters so that the other three recipients get wrong letters. Altogether, there are $20 \times 2 = 40$ ways to send the letters.

8. (10 points) Prove by mathematical induction that, for any $n \geq 2$,

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n-1}{n!} = 1 - \frac{1}{n!}.$$

Proof. (i) When $n = 2$, $l.h.s. = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2!} = r.h.s.$

(ii) Assume that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{m-1}{m!} = 1 - \frac{1}{m!}$. For $m + 1$, we have

$$\begin{aligned}
l.h.s. &= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{m-1}{m!} + \frac{(m+1)-1}{(m+1)!} \\
&= 1 - \frac{1}{m!} + \frac{(m+1)-1}{(m+1)!} \\
&= 1 - \frac{1}{m!} + \frac{(m+1)}{(m+1)!} - \frac{1}{(m+1)!} \\
&= 1 - \frac{1}{m!} + \frac{1}{m!} - \frac{1}{(m+1)!} \\
&= 1 - \frac{1}{(m+1)!} \\
&= r.h.s.
\end{aligned}$$

(i) and (ii) implies that

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n-1}{n!} = 1 - \frac{1}{n!},$$

for any $n \geq 2$. □