SOLUTIONS TO FINAL EXAM
MATH 455 SPRING 2004

1. *(15 pts)* Count the number of ways 33 players can be split into three soccer teams: Spartak, Dinamo, and Torpedo. (Note: Each soccer team has 11 players.)

   There are $\binom{33}{11}$ ways to pick players for Spartak and $\binom{22}{11}$ ways to pick players for Dinamo from the remaining 22. The rest plays for Torpedo. By multiplication rule there are $\binom{33}{11} \cdot \binom{22}{11}$ ways total.

2. *(15 pts)* Let $f : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be defined by the formula $f(m,n) = 2^m 5^n$.
   (a) *(10 pts)* Is f one-to-one?

   Yes, f is one-to-one. Indeed, every natural number has a unique decomposition into the product of primes. Thus if $2^m 5^n = 2^{m'} 5^{n'}$ then $m = m'$ and $n = n'$. More directly: $2^m 5^n = 2^{m'} 5^{n'}$ implies $2^{m-m'} = 5^{n'-n}$ which is only possible when $m - m'$ and $n' - n$ are both zero, i.e. $m = m'$ and $n = n'$.

   (b) *(5 pts)* Is f onto?

   No, f is not onto. Clearly not every natural number has only 2 and 5 as prime factors (e.g. 3 cannot be equal to $2^m 5^n$ for any natural m, n).

3. *(15 pts)* Let B_4 denote the set of all length 4 binary strings. Consider the Hamming distance function $Hd : B_4 \times B_4 \to \mathbf{Z}$.
   (a) *(5 pts)* What is the number of preimages of 0 under Hd?

   The distance between two strings is zero if and only if they are the same. Since there are 16 different strings in B_4, we get 16 preimages: $\{(u,u) \mid u \in B_4\}$.

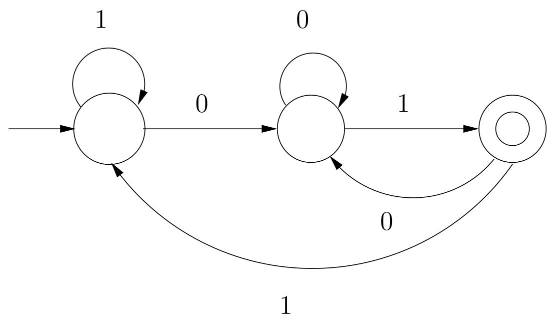
   (b) *(5 pts)* What is the number of preimages of 1 under Hd?

   Let u be any string in B_4. By flipping one of the digits in u we obtain a string u' of distance exactly one from u. Since there are 4 digits in u we get 4 different u' of distance one from u. Since u can be chosen in 16 ways we get $16 \cdot 4 = 64$ preimages.

   (c) *(5 pts)* What is the number of preimages of 5 under Hd?

   The distance between two strings of length 4 is at most 4. Therefore there are no preimages of 5.

4. *(15 pts)* Draw a finite state automaton for the input alphabet $\{0,1\}$ that accepts the set of all strings that end in 01. Clearly indicate the initial and accepting states.



Some of you had similar pictures with four states, which were also correct.

5. *(15 pts)* Consider the error correcting code $C = \{00000, 01011, 10101, 11110\}$. This is a $(5,4,d)$ error correcting code.
   (a) *(5 pts)* Is C linear?

   It is clear that adding 00000 to any code word does not change it. Thus you only need to check that $01011 + 10101 = 11110$, $01011 + 11110 = 10101$, and $10101 + 11110 = 01011$, which is true. (Remember that we add the strings *entry-wise* modulo two.) The code is linear.

   (b) *(5 pts)* Find d. How many errors can C correct? Explain.

   Since the code is linear it is enough to notice that there are at most 3 ones in every non-zero code word. Thus the minimal distance is $d = 3$. This implies that C can correct 1 error ($d = 2 \cdot 1 + 1$).

   (c) *(5 pts)* Is C perfect?

   Let's see how many strings we can correct: $4 + 4 \cdot 5 = 24$ (four code words and five corrupted code words per code word). Since the total number of 5-bit strings is $2^5 = 32$ not every string can be decoded, i.e. C is not perfect.

6. *(20 pts)* Let G be a simple graph with n vertices.
   (a) *(10 pts)* Prove that the number of edges of G is at most $n(n-1)/2$.

   Since G is simple, no edges begin and end at the same point and any two vertices are connected with at most one edge. Therefore, the simple graph with the most number of edges is the complete graph K_n where every pair of vertices is connected with an edge. Since there are $\binom{n}{2} = n(n-1)/2$ pairs the graph has $n(n-1)/2$ edges. Another (but similar) approach is to notice that the degree of each vertex in G is at most $n-1$, hence, the total degree is at most $n(n-1)$. By a theorem proved in class the number of edges is half the total degree and hence is at most $n(n-1)/2$. It is also not hard to give a proof by induction.

(b) *(10 pts)* Is it possible that the degrees of the vertices of G are all different? Justify.

Again, the degree of each vertex is at most $n-1$. If the degrees are all different it means that the degrees are $0, 1, 2, \ldots, n-1$. But this is impossible: if G has a vertex of degree $n-1$ then every other vertex is connected to it, thus no vertex can have degree 0.

7. *(15 pts)* Recall that a simple graph is *bipartite* if it is possible to color its vertices into two different colors in such a way that no two vertices of the same color are connected with an edge. Prove that a graph is bipartite if and only if it does not contain circuits with an odd number of edges.

First, suppose G is bipartite, and C is a circuit in G. Since the colors of the vertices in C must alternate it is impossible for C to have an odd number of vertices, and hence, edges.

Before we show the other implication let us prove a simple lemma:

Lemma. Any closed walk of odd length contains a circuit with an odd number of edges.

The proof is by induction on the number of edges. The base case is trivial (a closed walk of length 3 is a circuit). If the walk is a circuit we are done, otherwise it contains an edge repeated twice. After removing this edge we obtain a union of closed walks. Since we removed a repeated edge the total length of the obtained walks is still odd, hence, at least one of the walks will have odd length. By induction it will contain a circuit with an odd number of edges.

Now suppose G has no circuits with an odd number of edges. Then G has no walks of odd length by the above lemma. We are going to color vertices into two colors such that no two vertices of the same color are connected with an edge. Clearly it is enough to do this in every connected component of G. Pick any vertex u and color it black. For any other vertex v in the same connected component with u there is a path P_{uv} connecting u to v. Color the vertex v black if the number of edges in P_{uv} is even, otherwise color it white. The question is why the color of v is independent of the choice of the path P_{uv}. Indeed, if P'_{uv} is another path connecting u to v then the union of the two paths is a closed walk. By assumption it has an even length, therefore the number of edges of P_{uv} and of P'_{uv} have the same parity. This shows that the color of v is well-defined. The coloring we have constructed satisfies the definition. Indeed, if two vertices v and v' are connected with an edge e they must be of different colors, otherwise there would be paths P_{uv} and $P_{uv'}$ of the same parity and so their union along with the edge e would form a closed walk of odd length, which is prohibited.