DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MATH 455 May 17, 2004
FINAL EXAM, DURATION 2 HOURS

Your Name: ________________________________

Your Section: ____________________________

This exam paper consists of 7 questions. It has 6 pages. Show your work: all your answers must be justified. No calculators, books or notes are allowed!

1. (15) __________
2. (15) __________
3. (15) __________
4. (15) __________
5. (15) __________
6. (20) __________
7. (15) __________

TOTAL (110)
1. (15 pts) Count the number of ways 33 players can be split into three soccer teams: Spartak, Dinamo, and Torpedo. (Note: Each soccer team has 11 players.)

2. (15 pts) Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by the formula $f(m, n) = 2^m 5^n$.

   (a) (10 pts) Is $f$ one-to-one?

   (b) (5 pts) Is $f$ onto?
3. (15 pts) Let $B_4$ denote the set of all length 4 binary strings. Consider the Hamming distance function $Hd : B_4 \times B_4 \rightarrow \mathbb{Z}$.

(a) (5 pts) What is the number of preimages of 0 under $Hd$?

(b) (5 pts) What is the number of preimages of 1 under $Hd$?

(c) (5 pts) What is the number of preimages of 5 under $Hd$?
4. (15 pts) Draw a finite state automaton for the input alphabet \( \{0, 1\} \) that accepts the set of all strings that end in 01. Clearly indicate the initial and accepting states.

5. (15 pts) Consider the error correcting code \( C = \{00000, 01011, 10101, 11110\} \). This is a \((5, 4, d)\) error correcting code.

   (a) (5 pts) Is \( C \) linear?
   
   (b) (5 pts) Find \( d \). How many errors can \( C \) correct? Explain.
   
   (c) (5 pts) Is \( C \) perfect?
6. (20 pts) Let $G$ be a simple graph with $n$ vertices.

(a) (10 pts) Prove that the number of edges of $G$ is at most $n(n - 1)/2$.

(b) (10 pts) Is it possible that the degrees of the vertices of $G$ are all different? Justify.
7. (15 pts) Recall that a simple graph is bipartite if it is possible to color its vertices into two different colors in such a way that no two vertices of the same color are connected with an edge. Prove that a graph is bipartite if and only if it does not contain circuits with an odd number of edges.