Math 300

Metric Spaces.

Def. A metric space is a pair (M, d), where M is a non-empty set, and $d: M \times M \to [0, \infty)$ satisfies

(i) $d(p,q) = 0 \Leftrightarrow p = q$

(ii) $d(p,q) = d(q,p)$

(iii) $d(p,q) \leq d(p,w) + d(w,q)$ (Triangle Inequality)

Def. (M,d) is a metric space, and $r>0$, $p \in M$,
$B_r(p) = \{ q \in M \mid d(p,q) < r \}$.

Def. $U \subset M$ is an open set if $\forall p \in U$, $\exists r>0$, s.t., $B_r(p) \subset U$.

THM. $B_r(p)$ is an open set.
Proof. $q \in B_r(p) \Rightarrow d(q,p) < r$. Let $\varepsilon = r - d(p,q) > 0$.
If $y \in B_\varepsilon(q)$, then $d(p,y) \leq d(p,q) + d(q,y) < d(p,q) + \varepsilon < r$.
$\Rightarrow y \in B_r(p)$. $\Rightarrow B_\varepsilon(q) \subset B_r(p)$. $\Rightarrow B_r(p)$ is open.

THM. Assume (M,d) is a metric space. Then:
(i) M, $\emptyset$ are open
(ii) The intersection of a finite family of open sets is open.
(iii) The union of any family of open sets is open.

Def. $B \subset M$ is closed if $M - B$ is open.

THM. (i) $M, \emptyset$ are closed.

(ii) The union of a finite family of closed sets is closed.

(iii) The intersection of any family of closed sets is closed.

Proof. Use De Morgan's Laws:

$$\bigcup_{\alpha\in I}(M-A_\alpha) = M-\Big(\bigcap_{\alpha\in I} A_\alpha\Big), \quad \bigcap_{\alpha\in I}(M-A_\alpha) = M-\Big(\bigcup_{\alpha\in I} A_\alpha\Big).$$

Eg. $M=\mathbb{R}^2$, $d_1((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$(d_3 \le d_1 \le d_2 \le 2d_3)$ $\quad d_2((x_1,y_1),(x_2,y_2)) = |x_1-x_2|+|y_1-y_2|$

$d_3((x_1,y_1),(x_2,y_2)) = \max\{|x_1-x_2|, |y_1-y_2|\}$.

These are all metrics on $\mathbb{R}^2$. What is $B_r(p)$ under d_1, d_2, d_3?

Eg. $M=\mathbb{R}^2$, $d_4(p,q) = \begin{cases} 1 & p\neq q \\ 0 & p=q \end{cases}$. This is a metric on $\mathbb{R}^2$. What is $B_r(p)$ under d_4?

Def. $f:(M,d) \to (M',d')$ is continuous if, for any $p\in M$, and any $\varepsilon>0$, $\exists\ \delta>0$, s.t., $d'(f(p),f(q))<\varepsilon$ whenever $d(p,q)<\delta$.

Eg. $f:(\mathbb{R}^2,d_i)\to(\mathbb{R}^2,d_i)$ is continuous iff $f:(\mathbb{R}^2,d_j)\to(\mathbb{R}^2,d_j)$ is continuous. $i,j=1,2,3$.

When is $f:(\mathbb{R}^2,d_4)\to(\mathbb{R}^2,d_4)$ continuous?

Topological space.

Def. A Topological space is a pair $(M, \mathcal{T})$, s.t., $M \neq \emptyset$, $\mathcal{T} \in \mathcal{P}(M)$ satisfying

(i) $M, \emptyset \in \mathcal{T}$,

(ii) $\mathcal{F} \subset \mathcal{T}$, $|\mathcal{F}| < \infty \Rightarrow \bigcap_{A \in \mathcal{F}} A \in \mathcal{T}$.

(iii) $\mathcal{F} \subset \mathcal{T} \Rightarrow \bigcup_{A \in \mathcal{F}} A \in \mathcal{T}$.

$\mathcal{T}$ is called topology, and an element of $\mathcal{T}$ is called an open set of $(M, \mathcal{T})$. (A is closed if $M - A \in \mathcal{T}$).

Eg. $(M, \{M, \emptyset\})$, $(M, \mathcal{P}(M))$ are topological spaces.

Eg. If (M, d) is a metric space, and $\mathcal{T}_d = \{A \subset M \mid A \text{ is open with respect to } d\}$, then $(M, \mathcal{T}_d)$ is a topological space.

THM. If d_1, d_2 are two metrics on M, and $\exists\, c_1, c_2 > 0$ s.t., $c_1 d_1(p, q) \le d_2(p, q) \le c_2 d_1(p, q)$ $\forall p, q \in M$, then $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$.

Proof. Let $U \in \mathcal{T}_{d_1}$, and $p \in U$. Then $\exists\, r > 0$, s.t., $B_{r, d_1}(p) \subset U$. Consider $B_{c_1 r, d_2}(p)$. If $q \in B_{c_1 r, d_2}(p)$, the $d_1(p, q) \le \frac{1}{c_1} d_2(p, q) < \frac{1}{c_1} \cdot c_1 r = r \Rightarrow q \in B_{r, d_1}(p)$.
$\Rightarrow B_{c_1 r, d_2}(p) \subset B_{r, d_1}(p) \subset U$. $\Rightarrow U \in \mathcal{T}_{d_2}$.

Let $U \in \mathcal{T}_{d_2}$, and $p \in U$. Then $\exists\, r > 0$, s.t., $B_{r, d_2}(p) \subset U$. Consider $B_{\frac{r}{c_2}, d_1}(p)$. If $q \in B_{\frac{r}{c_2}, d_1}(p)$, then $d_2(p, q) \le c_2 d_1(p, q) < c_2 \cdot \frac{r}{c_2} = r \Rightarrow q \in B_{r, d_2}(p)$
$\Rightarrow B_{\frac{r}{c_2}, d_1}(p) \subset B_{r, d_2}(p) \subset U$. $\Rightarrow U \in \mathcal{T}_{d_1}$.

Eg. $M = \mathbb{R}^2$. d_i, $i = 1, 2, 3, 4$, are given as in page 2.
Then $\mathcal{T}_{d_1} = \mathcal{T}_{d_2} = \mathcal{T}_{d_3}$. What is $\mathcal{T}_{d_4}$?

$(M, \mathcal{T})$ is a topological space. $A \subset M$.

Def. (i) p is an interior point of A if $\exists\, U \in \mathcal{T}$, s.t., $p \in U \subset A$. $\mathrm{Int}(A) := \{p \mid p$ is an interior point of $A\}$

(ii) $\bar{A} := M - \mathrm{Int}(M-A)$. (Closure of A)

(iii) $\partial A := \bar{A} - \mathrm{Int}(A)$, a point in ∂A is called a boundary point of A.

THM. (i) $\mathrm{Int}\, A$ is the largest open set in A, i.e., $\mathrm{Int} A \subset A$, $\mathrm{Int} A \in \mathcal{T}$, and $B \subset \mathrm{Int} A$ if $B \in \mathcal{T}$, $B \subset A$.

(ii) $\bar{A}$ is the smallest closed set containing A, i.e., $A \subset \bar{A}$, $\bar{A}$ is closed, and $\bar{A} \subset B$ if B is closed and $A \subset B$.

Proof. (i) If $p \in \mathrm{Int}\, A$, then $p \in U \subset A$ for some $U \in \mathcal{T}$. So $p \in A$. $\Rightarrow \mathrm{Int} A \subset A$.

If $p \in \mathrm{Int}\, A$, then $\exists\, U \in \mathcal{T}$, s.t., $p \in U \subset A$. But, for any $q \in U$, we have $q \in U \subset A$. So $q \in \mathrm{Int}(A)$.
$\Rightarrow U \subset \mathrm{Int} A$. $\Rightarrow \mathrm{Int}\, A$ is open.

If $B \in \mathcal{T}$, $B \subset A$, then, for any $p \in B$, $p \in B \subset A$. $\Rightarrow p \in \mathrm{Int}\, A \Rightarrow B \subset \mathrm{Int}\, A$.

(ii) $\mathrm{Int}(M-A) \subset M-A$, $\mathrm{Int}(M-A) \in \mathcal{T}$, $\Rightarrow$
$A \subset \bar{A} = M - \mathrm{Int}(M-A)$, and $\bar{A}$ is closed.

If B is closed, and $A \subset B$, then
$M - B \in \mathcal{T}$, and $M-B \subset M-A$. $\Rightarrow M-B \subset \mathrm{Int}(M-A)$
$\Rightarrow B = M-(M-B) \supset M - \mathrm{Int}(M-A) = \bar{A}$.

THM. $A \subset M$. (i) A is open $\Leftrightarrow$ $A = \text{Int}\, A$

(ii) A is closed $\Leftrightarrow$ $A = \bar{A}$.

Proof. (i) "$\Leftarrow$" If $A = \text{Int}\, A$, then A is open since $\text{Int}\, A$ is open.

"$\Rightarrow$" If A is open, then $A \subset \text{Int}\, A \subset A$. So $A = \text{Int}\, A$.

(ii) "$\Leftarrow$" If $A = \bar{A}$, then A is closed since $\bar{A}$ is closed.

"$\Rightarrow$" If A is closed, then $A \subset \bar{A} \subset A$. $\Rightarrow A = \bar{A}$.

Eg. Since $\mathbb{Q}$ is countable, we have a sequence $\{r_n\}_{n=1}^{\infty}$, s.t.

$\mathbb{Q} = \{r_1, r_2, \cdots\}$. Let $I_n = (r_n - \frac{1}{2^n}, r_n + \frac{1}{2^n}) \subset \mathbb{R}$.

This is an open interval in $\mathbb{R}$. $A = \bigcup_{n=1}^{\infty} I_n$ is also open in $\mathbb{R}$.

Note that the measure of $A \leq \sum_{n=1}^{\infty} 2 \cdot \frac{1}{2^n} = 2$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{R} - A$ has not interior points $\Rightarrow \bar{A} = \mathbb{R}$.

So the measure of $\partial A = \bar{A} - \text{Int}\, A = \mathbb{R} - A$ is ∞.

Def. M, N are topological spaces. $f : M \to N$ is continuous if, for any open set U in N, $f^{-1}(U) = \{x \in M \mid f(x) \in U\}$ is an open set in M.

$f : M \to N$ is called a homeomorphism if f is a bijection, and f, f^{-1} are both continuous.

THM. (M, d_1), (N, d_2) are metric spaces. $f : (M, d_1) \to (N, d_2)$ is continuous if and only if $f : (M, \mathcal{T}_{d_1}) \to (N, \mathcal{T}_{d_2})$ is open.