1. Define \( f(z) = \frac{1}{2}(e^{zi} + e^{-zi}) \).

(a) Prove by definition that \( f(z) \) is holomorphic.

(b) Prove that \( f(z) = \cos z \) if \( z \in \mathbb{R} \), and conclude that \( f(z) \) is the unique holomorphic continuation of \( \cos x \).

(c) Find the unique holomorphic continuation of \( \sin x \).

2. (a) Prove that, for any complex functions \( f(z) \) and \( g(z) \),

\[
\frac{\partial}{\partial z}(f(z) + g(z)) = \frac{\partial}{\partial z}(f(z)) + \frac{\partial}{\partial z}(g(z)),
\]

\[
\frac{\partial}{\partial z}(f(z)g(z)) = \frac{\partial}{\partial z}(f(z))g(z) + f(z)\frac{\partial}{\partial z}(g(z)).
\]

(b) Prove that any polynomial of \( z \) with complex coefficients is holomorphic.

(\textit{Hint. Induct on the degree of the polynomial.})

3. Prove that

\[
\sum_{l=0}^{n-1} e^{\frac{2\pi l i}{n}} = \begin{cases} 
  n, & \text{if } k = 0, \\
  0, & \text{if } k = 1, 2, \cdots, n - 1.
\end{cases}
\]

(\textit{Hint. This is a geometric series.})

4. (a) Prove that points \( z_1 \) and \( z_2 \) are mirror images of each other across the straightline \( B\overline{z} + Bz + c = 0 \), where \( B \in \mathbb{C}, c \in \mathbb{R} \), if and only if \( B\overline{z_1} + B\overline{z_2} + c = 0 \).

(b) Find the mirror image of the origin across the line \( (3 + 5i)\overline{z} + (3 - 5i)z + 7 = 0 \).