1. Define $f(z) = \frac{1}{2}(e^{zi} + e^{-zi})$.

(a) Prove by definition that $f(z)$ is holomorphic.

(b) Prove that $f(z) = \cos z$ if $z \in \mathbb{R}$, and conclude that $f(z)$ is the unique holomorphic continuation of $\cos x$.

(c) Find the unique holomorphic continuation of $\sin x$.

2. (a) Prove that, for any complex functions $f(z)$ and $g(z)$,

$$\begin{aligned} \frac{\partial}{\partial \overline{z}}(f(z) + g(z)) &= \frac{\partial}{\partial \overline{z}}(f(z)) + \frac{\partial}{\partial \overline{z}}(g(z)), \\ \frac{\partial}{\partial \overline{z}}(f(z)g(z)) &= \frac{\partial}{\partial \overline{z}}(f(z))g(z) + f(z)\frac{\partial}{\partial \overline{z}}(g(z)). \end{aligned}$$

(b) Prove that any polynomial of z with complex coefficients is holomorphic.
(*Hint.* Induct on the degree of the polynomial.)

3. Prove that

$$\sum_{l=0}^{n-1} e^{\frac{2kl\pi}{n}i} = \begin{cases} n, & \text{if } k = 0, \\ 0, & \text{if } k = 1, 2, \cdots, n-1. \end{cases}$$

(*Hint.* This is a geometric series.)

4. (a) Prove that points z_1 and z_2 are mirror images of each other across the straightline $B\overline{z} + \overline{B}z + c = 0$, where $B \in \mathbb{C}$, $c \in \mathbb{R}$, if and only if $B\overline{z}_1 + \overline{B}z_2 + c = 0$.
(b) Find the mirror image of the origin across the line $(3 + 5i)\overline{z} + (3 - 5i)z + 7 = 0$.