

Prove the following propositions.

1. For any  $z, w \in \mathbb{C}$ ,

$$||z| - |w|| \leq |z - w|.$$

2. For any  $z_1, \dots, z_n \in \mathbb{C}$ ,

$$\left| \sum_{k=1}^n z_k \right| \leq \sum_{k=1}^n |z_k|.$$

(*Hint.* Use mathematical induction on  $n$ .)

3. For any  $z, w \in \mathbb{C}$ ,

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2).$$

4. If  $n = p_1^2 + p_2^2$  and  $m = q_1^2 + q_2^2$ , where  $p_1, p_2, q_1, q_2 \in \mathbb{Z}$ , then there exist  $r_1, r_2 \in \mathbb{Z}$  such that  $mn = r_1^2 + r_2^2$ .

(*Hint.* Consider the squares of absolute values of certain complex numbers.)