MATH 300: FUNDAMENTAL CONCEPTS OF MATHEMATICS SYLLABUS

Credits: 3 Credits.

Note: Math 300 students are required to register also for the 1-credit co-seminar Math 391A, which is taught by an undergraduate TA. Students will be assigned a seminar time during the first week of class.

Prerequisites: Math 131 and 132. **Co-requisite**: Math 235.

Suggested Texts:

- Class notes by Professor Hajir (available upon request).
- Class notes by Professor Meeks (available upon request).
- **The Mathematical Method: A Transition to Advanced Mathematics,** by M. Eisenberg, Prentice Hall, 1996.
- **How to Read and Do Proofs** by D. Solow, Fourth Edition, Wiley, 2005.
- **An Introduction to Mathematical Thinking** by W. Gilbert and S. Vanstone, Pearson Prentice-Hall, 2005.

General Course Description: Math 300 is designed to help students make the transition from calculus courses to the more theoretical junior-senior level mathematics courses. The goal of this course is to help students learn the language of rigorous mathematics, as structured by definitions, axioms, and theorems. Students will be trained in how to read, understand, devise and communicate proofs of mathematical statements. A number of proof techniques (contrapositive, contradiction, and induction) will be emphasized. The material includes set theory (Cantor's notion of size for sets and gradations of infinity, maps between sets, equivalence relations, partitions of sets), and basic logic (truth tables, negation, quantifiers). The material also includes discussion of the integers, rational numbers, real numbers, and complex numbers. One or two topics chosen from number theory, group theory, and point-set topology will also be included according to the interests of the instructor.

I. Basic Set Theory and Logic (Required)

Definitions:

(1) Given sets A, B, define $A \cap B$, $A \cup B$, $A \times B$.
(2) Define what it means for $f\colon A \to B$ to be a 1-1 function (injective) or an onto function (surjective) or a bijective function.
(3) Define the image of a function $f\colon A \to B$.
(4) For a function $f\colon A \to B$ and $C \subset B$, define the inverse image $f^{-1}(C)$.
(5) Inverse map f^{-1}.
(6) Equivalence relation on a set.
(7) Countable set.
(8) Uncountable set.
(9) Partition of a set.
(10) The power set $\mathcal{P}(A)$ of a set A.
(11) $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

Statements:

(1) Inverse map f^{-1} exists (and is unique) $\Leftrightarrow$ f is bijective.
(2) Well ordering principle.
(3) Principle of mathematical induction.

(4) Cantor's theorem.
(5) Fundamental theorem of equivalence relations.
(6) Binomial Theorem (recommended)

Representative problems to solve and proofs of theorems:

(1) The composition of injective functions is injective.
(2) The composition of surjective functions is surjective.
(3) Write down the truth table for $(p \wedge q) \to p$.
(4) Prove $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ using the principle of mathematical induction.
(5) What is the power set of $\{1, 2, 3\}$?
(6) Cantor's theorem (there is no surjection $f : A \to \mathcal{P}(A)$).
(7) $\mathbb{Q}$ is a countable set.
(8) $\mathbb{R}$ is an uncountable set.
(9) Suppose $f(x) = 2^x : \mathbb{R} \to \mathbb{R}$. What is the image of f? What is $f^{-1}([-1, 4])$?
(10) If A, B, C are countable infinite sets, then $A \cup B \cup C$ is a countable set.
(11) Find the multiplicative inverse of the complex number $5 + 7i$.

II. Basic Number Theory

Definitions:

(1) Congruence modulo m
(2) Construction of $\mathbb{Z}_m$
(3) Greatest common divisor
(4) Prime numbers

Statements:

(1) Division Algorithm
(2) Euclidean Algorithm
(3) Factorization of integers into primes

Representative problems to solve and proofs of theorems:

(1) Find the remainder of 3^{100} when dividing by 4.
(2) Calculate the g.c.d. of 484 and 451 and write it as a linear combination of 484 and 451.
(3) Show that $\sqrt{5}$ is not rational.
(4) How many positive divisors does 180 have?
(5) When does a fraction $\frac{a}{b}$ have a terminating decimal expansion in $\mathbb{R}$?

III. Basic Group Theory

Definitions:

(1) A group G.
(2) A subgroup $H \subset G$.
(3) Order of an element.
(4) Abelian and cyclic groups. A generator of a group.
(5) Group homomorphism.
(6) Center of a group.
(7) Kernel of a group homomorphism.
(8) Isomorphism.

Statements:

(1) A finite cyclic group is isomorphic to $\mathbb{Z}_m$ for one choice of positive integer m.

Representative problems to solve and proofs of theorems:

(1) Cancellation law in a group.
(2) Uniqueness of identity in a group.
(3) Uniqueness of inverses in a group.
(4) List the generators of $\mathbb{Z}_8$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$.
(5) The intersection of subgroups is a subgroup.
(6) The center of a group is a subgroup.
(7) The image of a group homomorphism is a subgroup.
(8) The kernel of a group homomorphism is a subgroup.
(9) A group homomorphism is injective if and only if its kernel contains only the identity element.
(10) The composition of group homomorphisms is a group homomorphism.

IV. Basic Topology

Definitions:

(1) Metric space.
(2) The ball $B_r(p)$ in a metric space.
(3) Open set in a metric space.
(4) Topological space.
(5) Closed set in a topological space.
(6) Limit point of a subset of a topological space.
(7) The closure of a subset in a topological space.
(8) Connected topological space.
(9) Compact topological space.
(10) Finite intersection property for a collection of sets.
(11) Subspace topology.
(12) Continuous function between topological spaces.

Statements:

(1) De Morgan's Laws.
(2) Least-upper-bound property of $\mathbb{R}$.
(3) Heine-Borel theorem.

Representative problems to solve and proofs of theorems:

(1) In a metric space, an open ball is an open set.
(2) In a metric space, the finite intersection of open sets is an open set.
(3) In a metric space, the union of any collection of open sets is an open set.
(4) A subset of a topological space is closed if and only if it contains all of its limit points.
(5) The intersection of a collection of closed sets is a closed set.
(6) A closed subset of a compact space is compact in the subspace topology.
(7) The composition of continuous functions is continuous.
(8) The continuous image of a connected space is connected in the subspace topology.
(9) The continuous image of a compact space is compact in the subspace topology.