Math 300

Complex numbers.

$\sqrt{-1}$ is not a real number. But we need to use it in some cases. So define $i = \sqrt{-1}$ to be an object that is linearly independent to 1 over $\mathbb{R}$, i.e., (for $a,b \in \mathbb{R}$) $a \cdot 1 + b \cdot i = 0$ iff $a = b = 0$.

Let $\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$. We say $a = \mathrm{Re}(a+bi)$, $b = \mathrm{Im}(a+bi)$.

Def. $(a+bi) + (c+di) = (a+c) + (b+d)i$

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} \quad \text{for } a+bi \neq 0.$$

$(\mathrm{Re}(z+w) = \mathrm{Re}\, z + \mathrm{Re}\, w, \quad \mathrm{Im}(z+w) = \mathrm{Im}\, z + \mathrm{Im}\, w)$

THM. "+", "·" are associative and commutative, and "·" is distributive over "+". For $a+bi \neq 0$, $\left(\frac{1}{a+bi}\right) \cdot (a+bi) = 1$.

Def. For $c+di \neq 0$, $\frac{a+bi}{c+di} = (a+bi) \cdot \left(\frac{1}{c+di}\right) = \frac{(ac+bd) + (bc-ad)i}{a^2+b^2}$.

Def. For $z = a+bi$, $\bar{z} := a - bi$, $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2+b^2}$

Lemma $z = 0 \Leftrightarrow \bar{z} = 0 \Leftrightarrow |z| = 0$

Using this def, one can see that $z^{-1} = 1/z = \bar{z}/|z|^2$, $\bar{\bar{z}} = z$

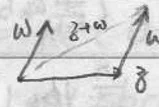
$$a = \mathrm{Re}\, z = (z+\bar{z})/2, \quad b = \mathrm{Im}\, z = (z-\bar{z})/2i, \quad |\mathrm{Re}\, z| \leq |z|, \quad |\mathrm{Im}\, z| \leq |z|$$

THM. $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$, $\overline{z/w} = \bar{z}/\bar{w}$ $(w \neq 0)$

$$|z \cdot w| = |z| \cdot |w|, \quad |z/w| = |z|/|w|, \quad w \neq 0.$$

THM. $|z+w| \leq |z| + |w|$

Proof.



Here we used the fact that $f: \mathbb{C} \to \mathbb{R}^2$, $f(a+bi) = (a,b)$ is a bijection, and the addition of complex numbers corresponds to vector addition under f. We shall not distinguish between $\mathbb{C}$ and $\mathbb{R}^2$.

That is every complex number $a+bi$ is represented by $\overset{\text{vector}}{\vec{}}(a,b)\in\mathbb{R}^2$, and vice versa.

Algebraic proof of the THM. $|z+w|^2 = (z+w)(\bar{z}+\bar{w}) = z\bar{z}+z\bar{w}+w\bar{z}+\bar{w}w$
$= |z|^2+|w|^2+2\mathrm{Re}(z\bar{w}) \le |z|^2+|w|^2+2|\mathrm{Re}(z\bar{w})| \le |z|^2+|w|^2+2|z|\cdot|\bar{w}|$
$= |z|^2+|w|^2+2|z||w| = (|z|+|w|)^2.$
$\Rightarrow \quad |z+w| \le |z|+|w|.$

Homework ① Prove by math induction that $\left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{n} |z_k|$

② Show $\big||z|-|w|\big| \le |z-w|$

③ Prove: if both m and n can be represented as sums of the squares of 2 integers, then $m\cdot n$ can also be represented as a sum of the squares of 2 integers.

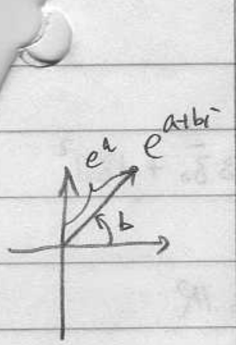
THM. For any $z_k, w_k \in \mathbb{C}$, $k=1,2,\dots,n$
$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \le \left(\sum_{k=1}^{n}|a_k|^2\right)\cdot\left(\sum_{k=1}^{n}|b_k|^2\right)$$

Proof. For any $t\in\mathbb{C}$, we have
$$0 \le \sum_{k=1}^{n}|a_k - t\bar{b}_k|^2 = \sum_{k=1}^{n}\left(|a_k|^2 - 2\,\mathrm{Re}(a_k\cdot\overline{t\bar{b}_k}) + |t\bar{b}_k|^2\right)$$
$$= \sum_{k=1}^{n}|a_k|^2 - 2\,\mathrm{Re}\left(\sum_{k=1}^{n}\bar{t}a_k b_k\right) + |t|^2\sum_{k=1}^{n}|b_k|^2$$
$$= \sum_{k=1}^{n}|a_k|^2 - 2\,\mathrm{Re}\left(\bar{t}\sum_{k=1}^{n}a_k b_k\right) + |t|^2\sum_{k=1}^{n}|b_k|^2$$

Specially, plug-in $t = \left(\sum_{k=1}^{n}a_k b_k\right)\Big/\left(\sum_{k=1}^{n}|b_k|^2\right)$.

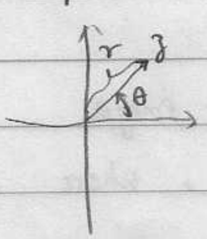
Then $0 \le \sum_{k=1}^{n}|a_k|^2 - 2\,\mathrm{Re}\left(\left(\overline{\sum_{k=1}^{n}a_k b_k}\right)\cdot\left(\sum_{k=1}^{n}a_k b_k\right)\right)\Big/\sum_{k=1}^{n}|b_k|^2$
$+ \dfrac{\left|\sum_{k=1}^{n}a_k b_k\right|^2}{\left(\sum_{k=1}^{n}|b_k|^2\right)^2}\cdot\sum_{k=1}^{n}|b_k|^2$, i.e.,

$$0 \le \sum_{k=1}^{n}|a_k|^2 - \frac{\left|\sum_{k=1}^{n}a_k b_k\right|^2}{\left(\sum_{k=1}^{n}|b_k|^2\right)} \Rightarrow \left|\sum_{k=1}^{n}a_k b_k\right| \le \left(\sum_{k=1}^{n}|a_k|^2\right)\left(\sum_{k=1}^{n}|b_k|^2\right).$$



More geometry. Define $e^{i\theta} = \cos\theta + i\sin\theta$ for $\forall\ \theta \in \mathbb{R}$

Then $e^{a+bi} = e^{a} \cdot e^{bi} = e^{a}(\cos b + i\sin b)$.



Any complex number $z = r(\cos\theta + i\sin\theta) = r \cdot e^{i\theta}$ for some $r, \theta \in \mathbb{R}$, $r \geq 0$. Clearly $|z| = r$.

Equation of a circle: $|z - z_0| = r$

z_0 = center, r = radius

Equation of a line: $ax + by + c = 0$, where $a, b, c \in \mathbb{R}$, (x, y) are the Cartesian coordinates of $\mathbb{R}^2$. Let $z = x + yi$, then $x = \operatorname{Re} z = \frac{1}{2}(z + \bar{z})$, $y = \operatorname{Im} z = \frac{1}{2i}(z - \bar{z}) = \frac{-i}{2}(z - \bar{z})$

$\Rightarrow$ Equation is $\frac{a}{2}(z + \bar{z}) - \frac{bi}{2}(z - \bar{z}) + c = 0$, i.e.,

$$\left(\frac{a}{2} - \frac{b}{2}i\right) z + \left(\frac{a}{2} + \frac{b}{2}i\right)\bar{z} + c = 0.$$

Let $B = \frac{a}{2} + \frac{b}{2}i$, then the equation is

$$\bar{B} z + B\bar{z} + c = 0, \quad B \in \mathbb{C}, \quad c \in \mathbb{R}.$$

THM. For $z, w \in \mathbb{C}$, $\operatorname{Re}(z \cdot \bar{w})$ = dot product of vectors corresponding to z, w.

Proof. $z = a + bi$, $w = c + di$, $z \rightsquigarrow (a, b)$, $w \rightsquigarrow (c, d)$.

$(a, b) \cdot (c, d) = ac + bd$.

$$\operatorname{Re}(z \cdot \bar{w}) = \operatorname{Re}((a+bi)(c-di)) = \operatorname{Re}((ac+bd) + (bc-ad)i)$$
$$= ac + bd$$

Cor. Vectors corresponding to z, w are $\perp$ iff $z \cdot \bar{w}$ is purely imaginary, i.e., $\operatorname{Re}(z \cdot \bar{w}) = 0$.

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More on the equation of a circle. $|z-z_0|=r$

$\Leftrightarrow |z-z_0|=r^2 \Leftrightarrow r^2=(z-z_0)(\bar{z}-\bar{z}_0)=|z|^2-z_0\bar{z}-z\bar{z}_0+|z_0|^2$.

$\rightsquigarrow$ Equation of a circle is of the form

$|z|^2+B\bar{z}+\bar{B}z+C=0$, where $B\in\mathbb{C}$, $C\in\mathbb{R}$.

Homework. ⓐ Prove that the points z_1 and z_2 are mirror images of each other across the straightline $B\bar{z}+\bar{B}z+C=0$, where $B\in\mathbb{C}\setminus\{0\}$, $C\in\mathbb{R}$, if and only if

$B\bar{z}_1+\bar{B}z_2+C=0$

ⓑ Prove that $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$

ⓑ Find the mirror image of the origin across the line $(3+5i)\bar{z}+(3-5i)z+7=0$.

More on $e^{i\theta}=\cos\theta+i\sin\theta$.

Using Tylor expansion $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$\cos x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$, $\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$

Plug $x=\theta i$ into e^x, we have

$$e^{i\theta}=\sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}=\sum_{k=0}^{\infty}\frac{(i\theta)^{2k}}{(2k)!}+\sum_{k=0}^{\infty}\frac{(i\theta)^{2k+1}}{(2k+1)!}$$

$(i^2=-1\Rightarrow i^{2k}=(-1)^k)$

$$=\sum_{k=0}^{\infty}(-1)^k\frac{\theta^{2k}}{(2k)!}+\sum_{k=0}^{\infty}(-1)^k\cdot i\,\frac{\theta^{2k+1}}{(2k+1)!}$$

$$=\cos\theta+i\sin\theta.$$

Question: How to continuate $\cos x$, $\sin x$ to a good complex function?

Complex partial derivatives:

Recall complex variables z & $\bar{z}$ are given by $z=x+iy$, $\bar{z}=x-iy$.

A complex function $f: A \to B$, (where A, B are open sets in $\mathbb{C}$,) is of the form $f(z) = u(x,y) + i\,v(x,y)$, where $u, v: \mathbb{R}^2 \to \mathbb{R}$ are real functions of 2 variables. We have $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ from Calc III. Define $\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$, $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$

In particular,
$$\frac{\partial}{\partial z}(f(z)) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) i$$
$$\frac{\partial}{\partial \bar{z}}(f(z)) = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) i$$

Def. $f: A \to B$ is called holomorphic if $\frac{\partial}{\partial \bar{z}}(f(z)) \equiv 0 \ \forall z \in A$.
$\left(\Leftrightarrow \ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ on } A\right)$

Eg. $e^z = e^{x+iy} = e^x(\cos y + i \sin y) = e^x \cos y + i e^x \sin y$
$$\frac{\partial}{\partial \bar{z}} e^z = \left(\frac{\partial}{\partial x}(e^x \cos y) - \frac{\partial}{\partial y}(e^x \sin y)\right) + i\left(\frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^x \cos y)\right)$$
$$= (e^x \cos y - e^x \cos y) + i(e^x \sin y - e^x \sin y) = 0$$
So e^z is holomorphic.

THM. If $f, g: \mathbb{C} \to \mathbb{C}$ are holomorphic functions, AB is a linesegment from $A \to B$, where $A \neq B$, $A, B \in \mathbb{C}$, and $f(z) = g(z) \ \forall z \in AB$, then $f(z) = g(z) \ \forall z \in \mathbb{C}$.

Remark. The reason for this is more or less that $\frac{\partial}{\partial \bar{z}}$ is an elliptic operator. You need to take complex analysis to understand the proof for the THM, and take advanced PDE to understand the properties of elliptic operators.

Homework. (i) Find the (unique) holomorphic extensions of $\cos x$, $\sin x$.
(ii) (a) Show that $\frac{\partial}{\partial \bar{z}}(f+g) = \frac{\partial}{\partial \bar{z}}(f) + \frac{\partial}{\partial \bar{z}}(g)$, $\frac{\partial}{\partial \bar{z}}(f \cdot g) = \left(\frac{\partial}{\partial \bar{z}}(f)\right) g + f \frac{\partial}{\partial \bar{z}}(g)$.
(b) Show that any polynomial of z is holomorphic.

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Complex numbers are introduced to solve the equation $x^2 = -1$.
Actually, they do a lot more.

(The Fundamental THM of Algebra)

THM. If $f(z)$ is a polynomial of degree n with complex coefficients, then $f(z) = c(z-r_1)(z-r_2)\cdots(z-r_n)$ for some $c, r_1, \cdots, r_n \in \mathbb{C}$.
That is, any polynomial equation over z is completly solvable.

Due to Abel's THM, we know that there is no way to find a formula for all these roots if $n \geq 5$. But we can still solve some special equations.

Eg. $x^n = 1$ has solutions $x_k = e^{\frac{2k\pi}{n} i}$, $k = 0, 1, 2, \cdots, n-1$.
(x_k is call a unit root of n-th order.)

Homework. Prove $\sum_{l=0}^{n-1} e^{\frac{2kl\pi}{n} i} = \begin{cases} n, & \text{if } k=0 \\ 0, & \text{if } k = 1, 2, \cdots, n-1. \end{cases}$

A model for complex numbers

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_{2\times2}(\mathbb{R})$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_{2\times2}(\mathbb{R})$.
Then $I^2 = I$, $J^2 = -I$, $IJ = JI = J$, and I, J are linearly independent.
Let $\mathcal{C} = \{ aI + bJ \mid a, b \in \mathbb{R} \}$. Define $f: \mathbb{C} \to \mathcal{C}$ by $f(a+bi) = aI + bJ$. Then f is a bijection, $f(z_1+z_2) = f(z_1) + f(z_2)$, $f(z_1 \cdot z_2) = f(z_1) \cdot f(z_2)$. So $\mathcal{C}$ and the matrix addition and multiplication on $\mathcal{C}$ realize $\mathbb{C}$ and its algebraic structure.

Final remark. Points and infinity: (i) Only one point at ∞. (S^2)
(ii) Every line has a point at ∞, and two line intersect at ∞ iff they are parallel. ($\mathbb{R}P^2$) (iii) Every line has two points at ∞. ($\mathbb{H}^2$).