The actual exam consists of two parts. In the first part, you will be asked to solve a combinatorial problem. In the second part of the exam, you will be asked to prove the following 3 theorems.

1. Prove by mathematical induction that, for any integer \( n \geq 1 \),
   \[
   \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.
   \]

2. Prove by mathematical induction that, for any integers \( n \geq r \geq 0 \),
   \[
   \sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}.
   \]

3. Prove that, for any integer \( n \geq 2 \),
   \[
   \left\lfloor \frac{n-1}{2} \right\rfloor \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (2k+1) \binom{n}{2k+1} = n \cdot 2^{n-2},
   \]
   where \( \left\lfloor \frac{n-1}{2} \right\rfloor \) is the greatest integer not greater than \( \frac{n-1}{2} \).