MATH 300 Practise Exam 1

October, 2006

The actual exam consists of two parts. In the first part, you will be asked to state 20 definitions/theorems from the notes. In some of these problems, instead of asking you to directly recite a definition/theorem, I will ask you to state or explain a general definition/theorem in a special case. In the second part of the exam, you will be asked to prove 5 theorems, all of which are selected from the theorems below.

1. Prove by the element argument that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

2. Prove that, if $f : A \to B$ and $g : B \to C$ are functions with $g \circ f : A \to C$ bijective, then f is injective and g is surjective.

3. (a) Prove that, if A and B are countable, then $A \times B$ is also countable.
(b) Prove by mathematical induction that, for any $n \geq 2$, $A_1 \times A_2 \times \cdots \times A_n$ is countable if $A_1, A_2, \cdots, A_n$ are all countable.

4. Prove that if $\mathcal{A} = \{A_n\}_{n \in \mathbb{N}}$, where A_n is countable $\forall\ n \in \mathbb{N}$, then $\cup \mathcal{A}$ is countable.

5. Prove Cantor's Theorem. That is, for any set A, $|A| < |\mathcal{P}(A)|$.

6. Prove by mathematical induction that $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$.

7. Prove that any subgroup of an abelian group is a normal subgroup.

8. Prove that, if G is a finite group and $g \in G$, then $g^{|G|} = e$, where e is the unit element of G.

9. Let G be a group. For $a \in G$, define $r_a : G \to G$ by $r_a(g) = aga^{-1}$. Define $\mathrm{Inner}(G) = \{r_a \mid a \in G\}$. Prove that $\mathrm{Inner}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

10. Define a binary operator "$\cdot$" on $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ by $(a + n\mathbb{Z}) \cdot (b + n\mathbb{Z}) = ab + n\mathbb{Z}$.
(a) Prove that "$\cdot$" is well defined.
(b) Prove that $(\mathbb{Z}_n - \{n\mathbb{Z}\}, \cdot)$ is a group when n is a prime number.
(c) Explain why $(\mathbb{Z}_n - \{n\mathbb{Z}\}, \cdot)$ is not a group when n is not a prime number.