The actual exam consists of one combinatorial problem and the following four problems.

1. Prove by mathematical induction that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n-1}{n!} = 1 - \frac{1}{n!}$ for all integers $n \geq 2$.

2. (a) State the definition of holomorphic functions.
(b) Define $f(z) = \frac{1}{2}(e^{zi} + e^{-zi})$. Prove that $f(z) = \cos z$ if $z \in \mathbb{R}$.
(c) Prove by definition that the function $f(z)$ from part (b) is holomorphic.

3. (M, d) is a metric space.
(a) State the definition of open sets in (M, d).
(b) Prove the following statements.
(1) M and $\emptyset$ are open.
(2) If $U_1, \cdots, U_n$ are open sets in M, then $\cap_{k=1}^{n} U_k$ is also open.
(3) If $\mathcal{F}$ is a family of open sets in M, then $\cup_{U \in \mathcal{F}} U$ is also open.

4. (a) State the definition of continuous functions from a metric space to a metric space.
(c) (M, d_1) and (N, d_2) are metric spaces. Prove that $f : (M, d_1) \to (N, d_2)$ is continuous if and only if the $f^{-1}(U)$ is an open subset of (M, d_1) whenever U is an open subset of (N, d_2).