Finite speed of propagation in porous media by mass transportation methods ⋆

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Abstract

In this note we make use of mass transportation techniques to give a simple proof of the finite speed of propagation of the solution to the one–dimensional porous medium equation. The result follows by showing that the difference of support of any two solutions corresponding to different compactly supported initial data is a bounded in time function of a suitable Monge–Kantorovich related metric. *To cite this article: J. A. Carrillo, M. P. Gualdani, G. Toscani, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

Résumé

Dans cette note nous utilisons des techniques de transport de masse pour donner une preuve élémentaire de la finitude de la vitesse de propagation des solutions de l'équation mono-dimensionnelle des milieux poreux. Le résultat repose sur la preuve de la propriété suivante : la différence du support entre deux solutions quelconques correspondant à des données initiales à support compact différentes est une fonction, bornée en temps, d'une métrique de Monge–Kantorovitch appropriée. *Pour citer cet article : J. A. Carrillo, M. P Gualdani, G. Toscani, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

1 Introduction

We consider the problem

$$u_t = (u^m)_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \quad m > 1, \tag{1}$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{2}$$

where $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $u_0 \geq 0$ and u_0 is compactly supported.

Much is already known for problem (1)-(2): see [1,2,3,4,5] and the references therein for existence, uniqueness and asymptotic behaviour results of the porous media equation. It also known that the degeneracy at level $u = 0$ of the diffusivity $D(u) = mu^{m-1}$ causes the phenomenon called *finite speed of propagation*. This means that the support of the solution $u(\cdot, t)$ to (1)-(2) is a bounded set for all $t \geq 0$. In fact it can be proved that the solution $u(x,t)$ as

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$t \to +\infty$ converges to the Barenblatt *source-type* solution $U(x,t,C)$ with the same mass as the initial data.

In this paper we want to give a simple proof of the *finite propagation* property using mass transportation techniques. Precisely, we prove that the difference of support of two solutions of (1)-(2) with different compactly supported initial conditions is a bounded in time function of a suitable Monge-Kantorovich related metric.

Theorem 1.1 *Let $u_1(x,t)$ and $u_2(x,t)$ be strong solutions of (1)-(2) with initial conditions $u_{01}(x)$ and $u_{02}(x)$ respectively, where $u_{0i} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $u_{0i} \geq 0$ and u_{0i} is compactly supported, $i = 1,2$, and let*

$$\Omega_i = \{(x,t) \in \mathbb{R} \times [0,+\infty)/u_i(x,t) > 0\}, \quad i = 1,2.$$

Let $\xi_i(t) = \inf_{x\in\mathbb{R}} \Omega_i$, $\Xi_i(t) = \sup_{x\in\mathbb{R}} \Omega_i$, for $t \geq 0$, $i = 1,2$. Then

$$\max\{|\xi_1(t) - \xi_2(t)|, |\Xi_1(t) - \Xi_2(t)|\} \leq W_\infty(u_{01}, u_{02}), \quad \forall t \in [0,+\infty), \tag{3}$$

where $W_\infty(u_{01}, u_{02})$ is a constant, which depends only on the initial data u_{01}, u_{02} and is defined in (18).

The finite speed of propagation property follows by just taking as one of the solutions a time translation of the explicit Barenblatt solution which is known to have compact support expanding at the rate $t^{1/(m+1)}$.

2 Proof

Consider a sequence of functions $u_n \in C^\infty([0,+\infty) \times \mathbb{R})$, which are strong solutions (see [3]) of the problems P_n

$$u_t = (u^m)_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \quad m > 1, \tag{4}$$
$$u(x,0) = u_{0n}(x), \quad x \in \mathbb{R}, \tag{5}$$

where $u_{0n}(x)$, $n \in \mathbb{N}$, is a sequence of bounded, integrable and strictly positive C^∞-smooth functions such that all their derivatives are bounded in $\mathbb{R}$, the condition $(m-1)(u_{0n}^m)_{xx} \geq -au_{0n}$ holds for some constant $a > 0$, and $u_{0n} \to u_0$ in $L^1(\mathbb{R})$ if $n \to +\infty$. We may always do it in such a way that $\|u_{0n}\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}$ and $\|u_{0n}\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}$. From the L^1-contraction property it follows that $u_n \to u$ in $C([0,+\infty) : L^1(\mathbb{R}))$ if $n \to +\infty$, where u is a strong solution of (1)-(2) (see [3], chapt. III).

This sequence of regularized solutions can be further approximated by a sequence of initial boundary value problems. We introduce a cutoff sequence $\theta_k \in C^\infty(\mathbb{R})$, $1 < k \in \mathbb{N}$, with the following properties:

$$\theta_k(x) = 1 \quad \text{for} \quad |x| < k-1, \tag{6}$$

$$\theta_k(x) = 0 \quad \text{for} \quad |x| \geq k, \quad 0 < \theta_k < 1 \quad \text{for} \quad k-1 < |x| < k. \tag{7}$$

The initial boundary value problem P_{nk}

$$u_t = (u^m)_{xx}, \quad x \in (-k,k), \quad t > 0, \tag{8}$$
$$u(x,0) = u_{0nk}(x) := \frac{u_{0n}(x)\theta_k(x)}{\|u_{0n}(x)\theta_k(x)\|_{L^1}}, \tag{9}$$
$$u(x,t) = 0 \quad \text{for} \quad |x| = k, \quad t \geq 0, \tag{10}$$

is mass preserving and has a unique solution $u_{nk}(x,t) \in C^\infty((0,+\infty) \times [-k,k]) \cap C([0,+\infty) \times [-k,k])$, strictly positive for $x \in (-k,k)$ and zero at the boundary (see [3], prop.6, chapt.II). Because $u_{0nk} \longrightarrow u_{0n}$ as $k \longrightarrow +\infty$, for all $n \in \mathbb{N}$, $u_{nk} \to u_n$ in $C([0,+\infty) : L^1(\mathbb{R}))$ if $k \to +\infty$, where u_n is solution of the problem P_n.

Thanks to estimates independent of k for the moments of the solutions of the P_{nk} problems and passing to the limit in the corresponding inequalities, it can be easily shown that the solution $u_n(x,t)$ of (4)-(5) enjoys an important property. It holds

$$\int_{\mathbb{R}} |x|^p u_n(x,t)dx < +\infty, \quad \forall t \geq 0, \quad \forall p \in [1,+\infty). \tag{11}$$

We shall denote by $\mathbb{P}_p(\mathbb{R})$, with $p \in [1,+\infty)$, the set of all probability measures on $\mathbb{R}$ with finite moments of order p. Let $\Pi(\mu,\nu)$ be the set of all probability measures on $\mathbb{R}^2$ having μ, $\nu \in \mathbb{P}_p(\mathbb{R})$ as marginal distributions (see [6]). The Wasserstein p-distance between two probability measures $\mu, \nu \in \mathbb{P}_p(\mathbb{R})$ is defined as

$$W_p(\mu,\nu)^p := \inf_{\pi\in\Pi(\nu,\mu)} \int_{\mathbb{R}^2} |x-y|^p d\pi(x,y), \quad \forall p \in [1,+\infty). \tag{12}$$

W_p defines a metric on $\mathbb{P}_p(\mathbb{R})$ (see [6]). Bound (11) then shows that the Wasserstein p-distance between any two solutions which is initially finite, remains finite at any subsequent time.

Any probability measure μ on the real line can be described in terms of its *cumulative distribution function* $F(x) = \mu((-\infty, x])$ which is a right-continuous and non-decreasing function with $F(-\infty) = 0$ and $F(+\infty) = 1$. Then, the *generalized inverse* of F defined by $F^{-1}(\eta) = \inf\{x \in \mathbb{R}/F(x) > \eta\}$ is also a right-continuous and non-decreasing function on $[0,1]$.

Let μ, $\nu \in \mathbb{P}_p(\mathbb{R})$ be probability measures and let $F(x)$, $G(x)$ be the respective distribution functions. On the real line (see [6]), the value of the Wasserstein p-distance $W_p(\mu,\nu)$ can be explicitly written in terms of the generalized inverse of the distribution functions,

$$W_p(\mu,\nu)^p = \int_0^1 |F^{-1}(\eta) - G^{-1}(\eta)|^p d\eta, \quad \forall p \in [1,+\infty). \tag{13}$$

Let $u_1(x,t)$, $u_2(x,t)$ be strong solutions of (1)-(2) corresponding to initial conditions $u_{01}(x)$ and $u_{02}(x)$ respectively. We denote by $u_{1n}(x,t)$ and $u_{2n}(x,t)$ the strong solutions of (4)-(5) with initial conditions $u_{01n}(x)$ and $u_{02n}(x)$ respectively, where $u_{0in} \longrightarrow u_{0i}$ in $L^1(\mathbb{R})$ for $i = 1,2$. Analogously, we consider the solutions $u_{1nk}(x,t)$ and $u_{2nk}(x,t)$ of the problems P_{nk} converging towards $u_{in}(x,t)$ for $i = 1,2$ in $C([0,+\infty) : L^1(\mathbb{R}))$ as $k \to \infty$.

Let $F_{ink}(x,t)$ be the distribution functions of u_{ink} for $i = 1,2$. A direct computation shows that $F_{ink}^{-1}(\eta,t)$ solves the following equation

$$\frac{\partial F_{ink}^{-1}}{\partial t} = -\frac{\partial}{\partial \eta}\left(\left(\frac{\partial F_{ink}^{-1}}{\partial \eta}\right)^{-m}\right), \quad i = 1,2, \tag{14}$$

for $t > 0$ and $\eta \in [0,1]$. Making use of equation (14), it is easy to prove that the Wasserstein p-distance

$$W_p(u_{1nk}, u_{2nk})(t) = \left\{\int_0^1 |F_{1nk}^{-1}(\eta,t) - F_{2nk}^{-1}(\eta,t)|^p d\eta\right\}^{\frac{1}{p}}, \quad \forall p \in [1,+\infty), \tag{15}$$

is a non-increasing in time function. In fact, for any given $p \geq 1$, integrating by parts one obtains

$$\frac{d}{dt}\int_0^1 |F_{1nk}^{-1}(\eta,t) - F_{2nk}^{-1}(\eta,t)|^p d\eta = p(p-1)\int_0^1 |F_{1nk}^{-1}(\eta,t) - F_{2nk}^{-1}(\eta,t)|^{p-2}$$
$$\times \left(F_{1nk}^{-1}(\eta,t)_\eta - F_{2nk}^{-1}(\eta,t)_\eta\right)\left[\left(F_{1nk}^{-1}(\eta,t)_\eta\right)^{-m} - \left(F_{2nk}^{-1}(\eta,t)_\eta\right)^{-m}\right] d\eta \leq 0$$

since the function x^{-m}, $m \geq 1$, is decreasing. Note that the boundary terms vanish due to the compact support of the solutions, which implies

$$\lim_{\eta\to 0^+}\left(F_{ink}^{-1}(\eta,t)_\eta\right)^{-1} = \lim_{\eta\to 1^-}\left(F_{ink}^{-1}(\eta,t)_\eta\right)^{-1} = 0 \qquad i = 1,2.$$

On the other hand, for all $p \in [1, +\infty)$,

$$W_p(u_{1nk}, u_{2nk}) \to W_p(u_{1n}, u_{2n}), \quad k \to +\infty, \tag{16}$$

$$W_p(u_{1n}, u_{2n}) \to W_p(u_1, u_2), \quad n \to +\infty. \tag{17}$$

This implies that $W_p(u_1, u_2) \leq W_p(u_{01}, u_{02})$, $\forall p \in [1, +\infty)$. Since the function $W_p(u_1, u_2)$ is increasing with respect to p, we can define the quantity

$$W_\infty(u_1, u_2) := \lim_{p \uparrow +\infty} W_p(u_1, u_2) = \sup_{\eta \in (0,1)} \mathrm{ess} |F_1^{-1}(\eta, t) - F_2^{-1}(\eta, t)|. \tag{18}$$

Since $W_\infty(u_{01}, u_{02})$ is finite, we deduce easily that $W_\infty(u_1, u_2)$ is also a non-increasing in time function.

Note that the inverse function $F^{-1}(\eta)$ of a distribution $F(x) = \int_{-\infty}^{x} u(s)ds$, where $u(s)$ is a integrable compactly supported function, is continuous at the point $\eta = 0$ and $\eta = 1$. Thus we can justify the inequality

$$\begin{gathered} W_\infty(u_1, u_2) = \sup_{\eta \in (0,1)} \mathrm{ess} |F_1^{-1}(\eta, t) - F_2^{-1}(\eta, t)| \geq \\ \max \left\{ |F_1^{-1}(0, t) - F_2^{-1}(0, t)|, |F_1^{-1}(1, t) - F_2^{-1}(1, t)| \right\} \geq \\ \max \left\{ |\xi_1(t) - \xi_2(t)|, |\Xi_1(t) - \Xi_2(t)| \right\} . \end{gathered} \tag{19}$$

We remark that the above arguments only hold in one space dimension due to the fact that only in this case one can express the p-Wasserstein distance in terms of pseudo-inverse distribution functions, as given in (13).

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