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A NONLINEAR FOURTH-ORDER PARABOLIC EQUATION WITH NONHOMOGENEOUS BOUNDARY CONDITIONS*

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Abstract. A nonlinear fourth-order parabolic equation with nonhomogeneous Dirichlet–Neumann boundary conditions in one space dimension is analyzed. This equation appears, for instance, in quantum semiconductor modeling. The existence and uniqueness of strictly positive classical solutions to the stationary problem are shown. Furthermore, the existence of global nonnegative weak solutions to the transient problem is proved. The proof is based on an exponential transformation of variables and new "entropy" estimates. Moreover, it is proved by the entropy–entropy production method that the transient solution converges exponentially fast to its steady state in the L^1 norm as time goes to infinity, under the condition that the logarithm of the steady state is concave. Numerical examples show that this condition seems to be purely technical.

Key words. fourth-order parabolic equation, fourth-order elliptic equation, existence and uniqueness of nonnegative solutions, entropy–entropy production method, exponential decay in time

AMS subject classifications. 35K30, 35K35, 35B40

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1. Introduction.

In recent years, the nonlinear fourth-order parabolic equation

$$u_t + (u(\log u)_{xx})_{xx} = 0, \quad u(\cdot,0) = u_I \geq 0 \quad \text{in } \Omega,\ t > 0, \tag{1.1}$$

in a bounded interval $\Omega = (0,1)$ with periodic or Dirichlet–Neumann boundary conditions or in the whole space $\Omega = \mathbb{R}$, has attracted the interest of many mathematicians since it possesses some interesting mathematical properties. For instance, the solutions are nonnegative, there are several Lyapunov functionals, and related logarithmic Sobolev inequalities can be derived [4, 10].

Equation (1.1) was first derived in the context of fluctuations of a stationary nonequilibrium interface [9]. It also appears as an approximation of the so-called quantum drift-diffusion model for semiconductors [1], which can be derived by a quantum moment method from a Wigner-BGK (Bhatnagar–Gross–Krook) equation [8]. More precisely, the quantum drift-diffusion model for the electron density u and the electron current density J reads as

$$u_t - J_x = 0, \quad J = -\frac{\varepsilon^2}{2} u \left(\frac{(\sqrt{u})_{xx}}{\sqrt{u}} \right)_x + T u_x + uE,$$

where ε is the scaled Planck constant, T the temperature, and $E = E(x,t)$ the electric field. Then (1.1) follows from this equation for $\varepsilon = 1$, zero temperature, and zero electric field since $u((\sqrt{u})_{xx}/\sqrt{u})_x = 2(u(\log u)_{xx})_x$.

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The first analytical result for (1.1) has been presented in [4]; more precisely, the existence of local-in-time positive classical solutions with periodic boundary conditions has been proved. This result has been generalized to global nonnegative weak solutions in [10]. The existence of global weak periodic solutions in several space dimensions has been proved very recently employing Wasserstein space techniques [12].

In quantum semiconductor modeling, Dirichlet–Neumann boundary conditions of the type

$$u(0,t) = u(1,t) = 1, \quad u_x(0,t) = u_x(1,t) = 0, \quad t > 0, \tag{1.2}$$

have been employed to model resonant tunneling diodes in $\Omega = (0,1)$ [14]. Here, the function $u(x,t)$ signifies the (nonnegative) electron density in the semiconductor device. The existence of global weak solutions to (1.1)–(1.2) has been proved in [13].

The boundary conditions (1.2) simplify the analysis considerably. Indeed, one of the main ideas of the existence proof is to employ an exponential transformation of variables, $u = e^y$. In the new variable y, the boundary conditions are homogeneous. Thus, using, for instance, the test function y in the weak formulation of (1.1), no integrals with boundary data appear.

The boundary conditions (1.2) follow from physical considerations like the charge neutrality at the boundary contacts, i.e., $u - C = 0$ at $x = 0, 1$, where $C = C(x)$ models fixed background charges. Numerical results show that the Neumann boundary conditions for the density u should be nonhomogeneous for ultrasmall semiconductor devices (see section 4 in [16]). Moreover, when the values of the doping profile $C(x)$ are different at the contacts, the Dirichlet boundary conditions satisfy $u(0,t) \neq u(1,t)$. Therefore, we wish to study the more general *nonhomogeneous* boundary conditions

$$u(0,t) = u_0, \quad u(1,t) = u_1, \quad u_x(0,t) = w_0, \quad u_x(1,t) = w_1, \quad t > 0, \tag{1.3}$$

where u_0, $u_1 > 0$ and w_0, $w_1 \in \mathbb{R}$. The treatment of the nonhomogeneities is also interesting from a mathematical point of view. Indeed, almost all results for (1.1) (and for related fourth-order equations like the thin-film model [3]) are shown only for periodic or no-flux boundary conditions or for whole-space problems, in order to avoid integrals with boundary data. In this paper, we show how to deal with nonhomogeneous boundary conditions for (1.1).

More precisely, we show (i) the existence and uniqueness of a classical positive solution u_∞ to the stationary problem corresponding to (1.1), (ii) the existence of global nonnegative weak solutions $u(\cdot,t)$ to the transient problem (1.1), (1.3), and (iii) the exponential convergence of $u(\cdot,t)$ to its steady state u_∞ as $t \to \infty$ in the L^1 norm, under the assumption that the boundary data is such that $\log u_\infty$ is concave. The long-time behavior is illustrated by numerical experiments. Notice that this is the first result of the *stationary* problem corresponding to (1.1) in the literature (if (1.2) or periodic boundary conditions are assumed, the steady state is constant). We also remark that the Wasserstein techniques of [12] cannot be applied to (1.1), (1.3) since this technique relies on the conservation of the L^1 norm which is not the case here.

The long-time behavior of solutions to (1.1) has been studied for periodic boundary conditions [5, 10] and for the boundary conditions (1.2) [15]. In particular, it could be shown that the solutions converge exponentially fast to their (constant) steady states. The decay rate has been numerically computed in [6]. No results are available up to now for the case of the nonhomogeneous boundary conditions (1.3).

Our first main result is the existence and uniqueness of stationary solutions needed in the existence proof for the transient problem.

THEOREM 1.1. *Let u_0, $u_1 > 0$ and w_0, $w_1 \in \mathbb{R}$. Then there exists a unique classical solution $u \in C^\infty([0,1])$ to*

$$(u(\log u)_{xx})_{xx} = 0 \quad \text{in } (0,1), \quad u(0) = u_0,\ u(1) = u_1, \qquad u_x(0) = w_0,\ u_x(1) = w_1, \tag{1.4}$$

satisfying $u(x) \geq m > 0$ for all $x \in [0,1]$, and the constant $m > 0$ depends only on the boundary data.

The existence proof is based on a fixed-point argument and appropriate a priori estimates, using the structure of the equation and the one-dimensionality heavily. More precisely, we perform the exponential transformation $u = e^y$ and write the equation in (1.4) as $y_{xx} = (ax+b)e^{-y}$ for some a, $b \in \mathbb{R}$. The key point is to derive uniform bounds on a and b. This implies a uniform H^1 bound for y and, in view of the one-dimensionality, a uniform L^∞ bound for $y = \log u$, hence showing the positivity of u. For the uniqueness we employ a monotonicity property of the operator $\sqrt{u} \mapsto -(u(\log u)_{xx})_{xx}/(2\sqrt{u})$ for suitable functions u (the monotonicity property was first observed in [13]).

The second main result is the existence of solutions to the transient problem. For simplicity, we consider time-independent boundary data only.

THEOREM 1.2. *Let u_0, $u_1 > 0$ and w_0, $w_1 \in \mathbb{R}$. Let $u_I(x) \geq 0$ be integrable such that $\int_0^1 (u_I - \log u_I)dx < \infty$. Then there exists a weak solution u to* (1.1), (1.3) *satisfying $u(x,t) \geq 0$ in $(0,1) \times (0,\infty)$ and*

$$u \in L^{5/2}_{\text{loc}}(0,\infty; W^{1,1}(0,1)) \cap W^{1,10/9}_{\text{loc}}(0,\infty; H^{-2}(0,1)),\ \log u \in L^2_{\text{loc}}(0,\infty; H^2(0,1)).$$

For the proof of this theorem we semidiscretize (1.1) in time and solve at each time step a nonlinear elliptic problem. The main difficulty is to obtain uniform estimates. The idea of [13] is to derive these estimates from a special Lyapunov functional,

$$E_1(t) = \int_0^1 \left(\frac{u}{u_\infty} - \log \frac{u}{u_\infty} \right) dx,$$

which is also called an "entropy" functional. Indeed, a formal computation (made precise in section 3) shows that

$$\frac{dE_1}{dt} + \int_0^1 (\log u)_{xx}^2 dx = \int_0^1 u(\log u)_{xx} \left(\frac{1}{u_\infty} \right)_{xx} dx, \tag{1.5}$$

implying that E_1 is nonincreasing if $(1/u_\infty)_{xx} = 0$, which is the case in [13] where $u_\infty = \text{const}$ holds. However, in the general case $(1/u_\infty)_{xx} \neq 0$, the right-hand side of (1.5) still needs to be estimated.

The key idea is to employ the *new* "entropy"

$$E_2(t) = \int_0^1 (\sqrt{u} - \sqrt{u_\infty})^2 dx.$$

A formal computation yields

$$\frac{dE_2}{dt} + 2\int_0^1 \left(\sqrt[4]{\frac{u_\infty}{u}} (\sqrt{u})_{xx} - \sqrt[4]{\frac{u}{u_\infty}} (\sqrt{u_\infty})_{xx} \right)^2 dx = 0. \tag{1.6}$$

With this estimate the right-hand side of (1.5) can be treated. Indeed, the above entropy production integral allows us to find the bound

$$\int_0^1 \left(\sqrt{u}(\log u)_{xx}^2 + (\sqrt[8]{u})_x^4\right) dx \le c \tag{1.7}$$

for some constant $c > 0$ depending only on the boundary data; see Lemma 3.2 for details. (Here and in the following, the notation $(f(u))_x^4$ means $[(f(u))_x]^4$.) Then, using Young's inequality, the right-hand side of (1.5) is bounded from above by

$$\int_0^1 \sqrt{u}(\log u)_{xx}^2 dx + \|1/u_\infty^2\|_{W^{2,\infty}(0,1)} \int_0^1 u^{3/2} dx,$$

which is bounded in view of (1.7). We stress the fact that this idea is new in the literature.

The above estimates are only valid if u is nonnegative. However, no maximum principle is generally available for fourth-order equations. We prove the nonnegativity property by employing the same idea as in the stationary case: after introducing an exponential variable $u = e^y$, we obtain a uniform H^2 bound by (1.5) and (1.7) and hence an L^∞ bound for $y = \log u$, which shows that u is positive. Letting the parameter of the time discretization tend to zero, we conclude the nonnegativity of u.

We notice that, interestingly, the new entropy E_2 is connected with the monotonicity property of $\sqrt{u} \mapsto -(u(\log u)_{xx})_{xx}/(2\sqrt{u})$ since the proof of this property also relies on the estimate (1.6) (see Lemma 2.3 in [13] and (2.7) below).

The physical (relative) entropy

$$E_3(t) = \int_0^1 \left(u \log \frac{u}{u_\infty} - u + u_\infty\right) dx$$

is still another Lyapunov functional. It is used in the proof of the long-time behavior of solutions, which is our final main result.

THEOREM 1.3. *Let the assumptions of Theorem* 1.2 *hold and let* $\int_0^1 u_I(\log u_I - 1)dx < \infty$. *Let* u *be the solution to* (1.1), (1.3) *constructed in Theorem* 1.2 *and let* u_∞ *be the unique solution to* (1.4). *We assume that the boundary data is such that* $\log u_\infty$ *is concave. Then there exist constants* $c, \lambda > 0$ *depending only on the boundary and initial data such that for all* $t > 0$,

$$\|u(\cdot,t) - u_\infty\|_{L^1(0,1)} \le ce^{-\lambda t}.$$

In order to prove this result, we take formally the time derivative of the relative entropy E_3. It can be shown (see section 4 for details) that the assumption $(\log u_\infty)_{xx} \le 0$ allows us to derive

$$\frac{dE_3}{dt} + P \le 0,$$

where $P \ge 0$ denotes the entropy production term involving second derivatives of u. This term can be estimated similarly as in [15] in terms of the entropy yielding

$$\frac{dE_3}{dt} + 2\lambda E_3 \le 0$$

for some $\lambda > 0$. Gronwall's inequality implies the exponential convergence in terms of the relative entropy. A Csiszar–Kullback-type inequality then gives the assertion. The assumption on the concavity of $\log u_\infty$ can be slightly relaxed (see Remark 4.4).

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. Then the existence of transient solutions (Theorem 1.2) is shown in section 3. Theorem 1.3 is proved in section 4, and finally in section 5, some numerical results are presented.

2. Existence and uniqueness of stationary solutions. In this section we will prove Theorem 1.1. First, we perform the transformation of variables $y = \log u$ and consider the problem

$$(e^y y_{xx})_{xx} = 0 \quad \text{in } (0,1), \quad y(0) = y_0,\ y(1) = y_1,\ y_x(0) = \alpha,\ y_x(1) = \beta, \tag{2.1}$$

where $y_0 = \log u_0$, $y_1 = \log u_1$, $\alpha = w_0/u_0$, and $\beta = w_1/u_1$. Clearly, any classical solution of (2.1) is a positive classical solution of (1.4). We show first some a priori estimates for the solution of (2.1).

LEMMA 2.1. *Let y be a classical solution to* (2.1). *Then*

$$y(x) \le M := \max\{y_0, y_1\} + |\alpha| + |\beta|. \tag{2.2}$$

Proof. First we observe that there exist constants a, $b \in \mathbb{R}$ such that y solves the equation $y_{xx} = (ax+b)e^{-y}$. This implies that y_{xx} can change its sign at most once. In the following we consider several cases for the sign of $y_{xx}(0)$ and $y_{xx}(1)$.

First case. Let $y_{xx}(0) \ge 0$ and $y_{xx}(1) \ge 0$. Since y_{xx} can change the sign at most once it follows that $y_{xx} \ge 0$ in $(0,1)$. We conclude that $y(x) \le \max\{y_0, y_1\}$ for all $x \in [0,1]$.

Second case. Let $y_{xx}(0) \ge 0$ and $y_{xx}(1) < 0$. There exists $x_1 \in [0,1)$ such that $y_{xx}(x_1) = 0$. Therefore, $ax+b \ge 0$ for all $x \in [0,x_1]$ and $ax+b \le 0$ for all $x \in [x_1,1]$. A Taylor expansion gives for all $x \in [x_1,1]$

$$\begin{aligned} y(x) &= y(1) + y_x(1)(x-1) + \int_x^1 (s-x) y_{xx}(s)\,ds \\ &= y_1 + \beta(x-1) + \int_x^1 (s-x)(as+b)e^{-y(s)}\,ds \le \max\{y_0, y_1\} + |\beta|. \end{aligned}$$

We claim that $y(x) \le \max\{y_0, y_1\} + |\beta|$ holds for all $x \in [0,x_1]$. For this, let $x_2 \in [0,x_1]$ be such that $y(x_2) = \max\{y(x) : x \in [0,x_1]\}$. Suppose that $y(x_2) > \max\{y_0, y_1\} + |\beta|$. Then $x_2 \in (0,x_1)$ and, since $y(x)$ reaches a maximum at the interior point x_2, $y_{xx}(x_2) \le 0$. Since $x_2 \in (0,x_1)$, we have $y_{xx}(x_2) = (ax_2+b)e^{-y(x_2)} \ge 0$. This shows that $y_{xx}(x_2) = 0$. But then $y_{xx}(x_2) = (ax_2+b)e^{-y(x_2)}$ implies that $ax_2+b = 0$. Since also $ax_1+b = 0$, it follows that $a = b = 0$ and thus $y_{xx}(x) = 0$ for all $x \in [0,1]$; this is a contradiction to $y_{xx}(1) < 1$. Hence, $y(x) \le \max\{y_0, y_1\} + |\beta|$ for all $x \in [0,1]$.

Third case. Let $y_{xx}(0) < 0$ and $y_{xx}(1) \ge 0$. By similar arguments as in the second case, it can be shown that $y(x) \le \max\{y_0, y_1\} + |\alpha|$ for all $x \in [0,1]$.

Fourth case. Let $y_{xx}(0) < 0$ and $y_{xx}(1) < 0$. This implies that $ax+b < 0$ for all $x \in [0,1]$ and, by a Taylor expansion,

$$y(x) = y_0 + \alpha x + \int_0^x (x-s)(as+b)e^{-y(s)}\,ds \le y_0 + |\alpha|, \quad x \in [0,1].$$

The lemma is proved. ∎

LEMMA 2.2. *Let y be a classical solution to* (2.1). *Then there exists a constant $K > 0$ depending only on y_0, y_1, α, and β such that*

$$\|y\|_{H^2(0,1)} \le K.$$

Proof. There exist constants a, $b \in \mathbb{R}$ such that y solves the equation

$$y_{xx} = (ax+b)e^{-y} \quad \text{in } (0,1), \tag{2.3}$$

and $b = e^{y_0} y_{xx}(0)$, $a = e^{y_1} y_{xx}(1) - e^{y_0} y_{xx}(0)$. In order to obtain a uniform estimate for y_{xx} we first have to find uniform estimates for a and b. For this, we multiply (2.3) by y_x^2 and integrate over $(0,1)$:

$$\int_0^1 (ax+b)e^{-y} y_x^2 dx = \int_0^1 y_{xx} y_x^2 dx = \frac{1}{3}\int_0^1 (y_x^3)_x dx = \frac{1}{3}(\beta^3 - \alpha^3).$$

Next we multiply (2.3) by y_{xx}, integrate over $(0,1)$, integrate by parts, and use the above equality:

$$\begin{aligned}\int_0^1 y_{xx}^2 dx &= \int_0^1 (ax+b)e^{-y} y_{xx} dx \\ &= \int_0^1 (ax+b)e^{-y} y_x^2 dx - a\int_0^1 e^{-y} y_x dx + [(ax+b)e^{-y(x)} y_x(x)]_0^1 \\ &= \frac{1}{3}(\beta^3 - \alpha^3) + a(e^{-y_1} - e^{-y_0}) + (a+b)e^{-y_1}\beta - be^{-y_0}\alpha.\end{aligned}$$

By Young's inequality this becomes

$$\int_0^1 y_{xx}^2 dx \le C + \frac{1}{60}e^{-2M}a^2 + \frac{1}{12}e^{-2M}b^2, \tag{2.4}$$

where $C := (\beta^3 - \alpha^3)/3 + 15e^{2M}((1+\beta)e^{-y_1} - e^{-y_0})^2 + 3e^{2M}(\beta e^{-y_1} - \alpha e^{-y_0})^2$. Taking the square of (2.3) and integrating over $(0,1)$ yields, by Lemma 2.1,

$$\begin{aligned}\int_0^1 y_{xx}^2 dx &= \int_0^1 (ax+b)^2 e^{-2y} dx \ge e^{-2M}\int_0^1 (ax+b)^2 dx \\ &= \frac{1}{3}e^{-2M}(a^2 + 3ab + 3b^2) \ge \frac{1}{3}e^{-2M}\left(\frac{a^2}{10} + \frac{b^2}{2}\right),\end{aligned} \tag{2.5}$$

where we have used the Young inequality $3ab \ge -9a^2/10 - 5b^2/2$. Putting together (2.4) and (2.5), we obtain

$$\frac{a^2}{10} + \frac{b^2}{2} \le 3e^{2M}\int_0^1 y_{xx}^2 dx \le 3e^{2M}C + \frac{a^2}{20} + \frac{b^2}{4}. \tag{2.6}$$

Therefore, a and b are bounded by a constant which depends only on y_0, y_1, α, and β. By (2.4) this gives a uniform estimate for $\|y_{xx}\|_{L^2(0,1)}$ and, employing Poincaré's inequality, also for $\|y\|_{H^2(0,1)}$. ∎

Proof of Theorem 1.1. We wish to employ the Leray–Schauder fixed-point theorem. For this let $\sigma \in [0,1]$ and $z \in H^1(0,1)$ and let $y \in H^2(0,1)$ be the unique solution of

$$(e^z y_{xx})_{xx} = 0 \quad \text{in } (0,1), \quad y(0) = \sigma y_0,\ y(1) = \sigma y_1,\ y_x(0) = \sigma\alpha,\ y_x(1) = \sigma\beta.$$

This defines a fixed-point operator $S : H^1(0,1) \times [0,1] \to H^1(0,1)$, $S(z,\sigma) = y$. Clearly, $S(z,0) = 0$ for all z. Moreover, by standard arguments, S is continuous and compact, since the embedding $H^2(0,1) \hookrightarrow H^1(0,1)$ is compact. It remains to show

that there exists a constant $K > 0$ such that for all $\sigma \in [0,1]$ and for any fixed point y of $S(\cdot,\sigma)$, the estimate $\|y\|_{H^1(0,1)} \le K$ holds. Lemma 2.2 settles the case $\sigma = 1$. For $\sigma < 1$, a similar proof as in Lemma 2.2 shows the existence of a constant $K > 0$ such that $\|y\|_{H^2(0,1)} \le K$ holds. By the Leray–Schauder theorem, this proves the existence of a solution $y \in H^2(0,1)$ to (2.1).

Actually, the solution y is a classical solution. Indeed, y satisfies $y_{xx} = (ax + b)e^{-y} \in H^2(0,1)$ for some a, $b \in \mathbb{R}$, and hence, $y \in H^4(0,1)$. By bootstrapping, $y \in H^n(0,1)$ for all $n \in \mathbb{N}$ and y is a classical solution.

In order to prove the uniqueness of solutions, we extend an idea of [13]. Let u_1 and u_2 be two positive classical solutions to (1.4). We multiply (1.4) for u_1 by $1 - \sqrt{u_2/u_1}$ and (1.4) for u_2 by $\sqrt{u_1/u_2} - 1$, integrate, and take the difference. This yields, by integrating by parts,

$$
\begin{aligned}
(2.7)\quad 0 &= \int_0^1 \Big[(u_1(\log u_1)_{xx})_{xx}(1 - \sqrt{u_2/u_1}) - (u_2(\log u_2)_{xx})_{xx}(\sqrt{u_1/u_2} - 1)\Big]\,dx \\
&= 2\int_0^1 \Big[(\sqrt{u_1})_{xxxx} - \frac{1}{\sqrt{u_1}}(\sqrt{u_1})_{xx}^2 - (\sqrt{u_2})_{xxxx} \\
&\qquad\qquad + \frac{1}{\sqrt{u_2}}(\sqrt{u_2})_{xx}^2\Big](\sqrt{u_1} - \sqrt{u_2})dx \\
&= 2\int_0^1 \Big[((\sqrt{u_1})_{xx} - (\sqrt{u_2})_{xx})(\sqrt{u_1} - \sqrt{u_2})_{xx} \\
&\quad - (\sqrt{u_1})_{xx}^2\left(1 - \sqrt{\tfrac{u_2}{u_1}}\right) + (\sqrt{u_2})_{xx}^2\left(\sqrt{\tfrac{u_1}{u_2}} - 1\right)\Big]dx \\
&= 2\int_0^1 \left(\sqrt[4]{\frac{u_2}{u_1}}(\sqrt{u_1})_{xx} - \sqrt[4]{\frac{u_1}{u_2}}(\sqrt{u_2})_{xx}\right)^2.
\end{aligned}
$$

Therefore,

$$
0 = \sqrt[4]{\frac{u_2}{u_1}}(\sqrt{u_1})_{xx} - \sqrt[4]{\frac{u_1}{u_2}}(\sqrt{u_2})_{xx} \quad \text{in } (0,1).
$$

Writing $u_1 = e^{y_1}$ and $u_2 = e^{y_2}$, this identity is equal to

$$
\begin{aligned}
0 &= e^{(y_2-y_1)/4}(e^{y_1/2})_{xx} - e^{(y_1-y_2)/4}(e^{y_2/2})_{xx} \\
&= \frac{1}{2}e^{(y_2+y_1)/4}\left(y_{1,xx} + \frac{1}{2}y_{1,x}^2\right) - \frac{1}{2}e^{(y_1+y_2)/4}\left(y_{2,xx} + \frac{1}{2}y_{2,x}^2\right),
\end{aligned}
$$

and hence

$$
(2.8)\qquad y_{1,xx} - y_{2,xx} = -\frac{1}{2}(y_{1,x}^2 - y_{2,x}^2) \quad \text{in } (0,1).
$$

We integrate this equation over $(0,x_0)$, use the boundary condition $y_{1x}(0) = y_{2x}(0)$, and take the supremum,

$$
\|(y_1 - y_2)_x\|_{L^\infty(0,x_0)} \le \int_0^{x_0} |(y_1 + y_2)_x| \cdot |(y_1 - y_2)_x| dx \le x_0 L \|(y_1 - y_2)_x\|_{L^\infty(0,x_0)},
$$

where $L = \|y_{1,x}\|_{L^\infty(0,1)} + \|y_{2,x}\|_{L^\infty(0,1)}$. Choosing $x_0 = 1/2L$ gives $(y_1 - y_2)_x = 0$ and hence $y_1 - y_2 = 0$ in $[0,x_0]$. In particular, $(y_1 - y_2)_x(x_0) = 0$. Therefore, integrating (2.8) over $(x_0, 2x_0)$ we obtain by the same arguments that $y_1 - y_2 = 0$ in $[x_0, 2x_0]$.

After a finite number of steps we achieve $y_1 - y_2 = 0$ in $[0,1]$. This proves the uniqueness of solutions. $\square$

Remark 2.3. Equation (2.3) with $y(0) = y(1)$ and $y_x(0) = -y_x(1) \le 0$ is formally related to a combustion problem. Indeed, the boundary conditions imply that y is symmetric around $x = \frac{1}{2}$ and that $y(x) \le y(0) = y_0$ holds for any $x \in [0,1]$. The symmetry implies further $a = e^{y_0}(y_{xx}(1) - y_{xx}(0)) = 0$ and moreover, $b = e^{y_0} y_{xx}(0) \ge 0$. Thus we can write (2.3) as $y_{xx} = be^{-y}$ or, introducing $z(x) = -y(x)$,

$$z_{xx} + be^z = 0 \quad \text{in } (0,1), \quad z(0) = z(1) = -y_0.$$

This is the solid fuel ignition model of [2]. It is well known that there exists $b^* > 0$ such that this problem has exactly two solutions if $b \in (0, b^*)$, it has exactly one solution if $b = b^*$, and it has no solution if $b > b^*$ [2, 11]. This relation provides a better bound for b (for the above special boundary conditions) than the estimate (2.6). Indeed, $a = 0$ and b is uniformly bounded by a number $b^* > 0$ *independently* of the boundary conditions (and depending only on the domain $(0,1)$).

3. Existence of transient solutions. In order to prove Theorem 1.2 we again perform the exponential change of unknowns and we semidiscretize (1.1) in time. For this, we divide the time interval $(0,T]$ for some $T > 0$ into N subintervals $(t_{k-1}, t_k]$, with $k = 1, \ldots, N$, where $0 = t_0 < \cdots < t_N = T$. Define $\tau_k = t_k - t_{k-1} > 0$ and $\tau = \max\{\tau_k : k = 1, \ldots, N\}$. We assume that $\tau \to 0$ as $N \to \infty$.

Let $u_\infty > 0$ be the unique classical solution to (1.4) and set $y_\infty = \log u_\infty$. We perform the transformation $z = \log(u/u_\infty)$ and $z_0 = \log(u_I/u_\infty)$. For given $k \in \{1, \ldots, N\}$ and z_{k-1} we first solve the semidiscrete problem

$$\frac{e^{y_\infty}}{\tau_k}(e^{z_k} - e^{z_{k-1}}) = -\left(e^{z_k + y_\infty}(z_k + y_\infty)_{xx}\right)_{xx}, \quad z_k \in H_0^2(0,1). \tag{3.1}$$

PROPOSITION 3.1. *For each $k = 1, \ldots, N$, there exists a unique weak solution $z_k \in H_0^2(0,1)$ to* (3.1).

For the proof of this proposition we first show some a priori estimates.

LEMMA 3.2. *Let $z_k \in H_0^2(0,1)$ be a weak solution to* (3.1). *Then there exists a constant $c > 0$ depending only on T, u_I, and u_∞ such that*

$$\|e^{z_k/2}\|_{L^2(0,1)} \le c, \tag{3.2}$$

$$\sum_{i=1}^N \tau_i \int_0^1 e^{z_i/2}\left((z_i + y_\infty)_{xx}^2 + (z_i + y_\infty)_x^4\right) \le c, \tag{3.3}$$

$$\sum_{i=1}^N \tau_i \|e^{z_i}\|_{L^\infty(0,1)} \le c. \tag{3.4}$$

Proof. Similarly as in the uniqueness proof of Theorem 1.1 we use the test functions $1 - e^{-z_k/2} \in H_0^2(0,1)$ in the weak formulation of the semidiscretized equation (3.1) and $e^{z_k/2} - 1 \in H_0^2(0,1)$ in the weak formulation of the stationary equation (1.4) and take the sum of the corresponding equations:

$$\begin{aligned}\frac{1}{\tau_k}\int_0^1 e^{y_\infty}(e^{z_k} - e^{z_{k-1}})(1 - e^{-z_k/2})dx &= \int_0^1 e^{z_k + y_\infty}(z_k + y_\infty)_{xx}(e^{-z_k/2})_{xx}dx \\ &\quad + \int_0^1 e^{y_\infty} y_{\infty,xx}(e^{z_k/2})_{xx}dx.\end{aligned} \tag{3.5}$$

The right-hand side is equal to the first integral in (2.7) with $u_1 = e^{z_k+y_\infty}$ and $u_2 = e^{y_\infty}$. Therefore, the right-hand side is equal to the expression

$$-2\int_0^1 \left(e^{-z_k/4}(e^{(z_k+y_\infty)/2})_{xx} - e^{z_k/4}(e^{y_\infty/2})_{xx}\right)^2 dx.$$

For the left-hand side of (3.5) we write

$$\begin{aligned}
&\frac{1}{\tau_k}\int_0^1 e^{y_\infty}(e^{z_k} - e^{z_{k-1}})(1 - e^{-z_k/2})dx \\
&= \frac{1}{\tau_k}\int_0^1 e^{y_\infty}((e^{z_k/2} - 1)^2 - (e^{z_{k-1}/2} - 1)^2)dx + \frac{1}{\tau_k}\int_0^1 e^{y_\infty}(e^{z_k/4} - e^{z_{k-1}/2 - z_k/4})^2 dx \\
&\geq \frac{1}{\tau_k}\int_0^1 e^{y_\infty}\left((e^{z_k/2} - 1)^2 - (e^{z_{k-1}/2} - 1)^2\right) dx.
\end{aligned}$$

Therefore, we conclude from (3.5), for all $k = 1, \dots, N$,

$$\begin{aligned}
&\frac{1}{\tau_k}\int_0^1 e^{y_\infty}(e^{z_k/2} - 1)^2 dx + 2\int_0^1 \left(e^{-z_k/4}(e^{(z_k+y_\infty)/2})_{xx} - e^{z_k/4}(e^{y_\infty/2})_{xx}\right)^2 dx \\
&\qquad \leq \frac{1}{\tau_k}\int_0^1 e^{y_\infty}(e^{z_{k-1}/2} - 1)^2 dx. \qquad (3.6)
\end{aligned}$$

This yields

$$\int_0^1 e^{y_\infty}(e^{z_k/2} - 1)^2 dx \leq \int_0^1 e^{y_\infty}(e^{z_0/2} - 1)^2 dx = \int_0^1 (\sqrt{u_I} - \sqrt{u_\infty})^2 dx < \infty \qquad (3.7)$$

and thus (3.2). Moreover, after summing up (3.6),

$$2\sum_{i=1}^k \tau_i \int_0^1 \left(e^{-z_i/4}(e^{(z_i+y_\infty)/2})_{xx} - e^{z_i/4}(e^{y_\infty/2})_{xx}\right)^2 dx \leq \int_0^1 e^{y_\infty}(e^{z_0/2} - 1)^2 dx.$$

Young's inequality gives

$$4\sum_{i=1}^k \tau_i \int_0^1 e^{-z_i/2}\left(e^{(z_i+y_\infty)/2}\right)_{xx}^2 dx \leq c + c\sum_{i=1}^k \tau_i \int_0^1 e^{z_i/2} dx,$$

where here and in the following, $c > 0$ denotes a generic constant depending only on T, u_I, and u_∞. In view of (3.7), the right-hand side is uniformly bounded. Hence

$$\begin{aligned}
&\sum_{i=1}^k \tau_i \int_0^1 e^{-(z_i+y_\infty)/2}\left(e^{(z_i+y_\infty)/2}\right)_{xx}^2 dx \\
&\leq \|e^{y_\infty/2}\|_{L^\infty(0,1)} \sum_{i=1}^k \tau_i \int_0^1 e^{-z_i/2}\left(e^{(z_i+y_\infty)/2}\right)_{xx}^2 dx \leq c.
\end{aligned}$$

Now the assertion (3.3) follows since, by integration by parts,

$$\int_0^1 e^{u/2} u_x^2 u_{xx} dx = -\frac{1}{6}\int_0^1 e^{u/2} u_x^4 + \frac{1}{3}(e^{u(1)/2} u_x(1)^3 - e^{u(0)/2} u_x(0)^3)$$

for all $u \in H^2(0,1)$, and hence,

$$\begin{aligned}
&\int_0^1 e^{-(z_i+y_\infty)/2}\left(e^{(z_i+y_\infty)/2}\right)_{xx}^2 dx \\
&= \frac{1}{4}\int_0^1 e^{(z_i+y_\infty)/2}\left((z_i+y_\infty)_{xx}^2 + \frac{1}{4}(z_i+y_\infty)_x^4 + (z_i+y_\infty)_{xx}(z_i+y_\infty)_x^2\right) dx \\
&= \frac{1}{4}\int_0^1 e^{(z_i+y_\infty)/2}\left((z_i+y_\infty)_{xx}^2 + \frac{1}{12}(z_i+y_\infty)_x^4\right) dx + \frac{1}{12}(e^{y_1/2}\beta^3 - e^{y_0/2}\alpha^3).
\end{aligned}$$

Finally, (3.4) is a consequence of (3.3) and the Poincaré–Sobolev inequality since

$$\int_0^1 e^{z_i/2}(z_i)_x^4 dx = 8^4\int_0^1 (e^{z_i/8})_x^4 \geq c\|e^{z_i/8}\|_{L^\infty(0,1)}^4.$$

This shows the lemma. ∎

LEMMA 3.3. *Let $z_k \in H_0^2(0,1)$ be a weak solution to* (3.1). *Then there exists a constant $c > 0$ depending only on T, u_I, and u_∞ such that*

$$\int_0^1 (e^{z_k} - z_k)dx + \sum_{i=1}^k \tau_i \int_0^1 (z_i + y_\infty)_{xx}^2 dx \leq c. \tag{3.8}$$

Proof. We choose the test function $e^{-y_\infty}(1-e^{-z_k}) \in H_0^2(0,1)$ in the weak formulation of (3.1). Then, by Young's inequality,

$$\begin{aligned}
&\int_0^1 (e^{z_k} - e^{z_{k-1}})(1-e^{-z_k})dx \\
&= -\tau_k\int_0^1 e^{z_k}(z_k+y_\infty)_{xx}(y_{\infty,x}^2 - y_{\infty,xx})dx - \tau_k\int_0^1 (z_k+y_\infty)_{xx}^2 dx \\
&\quad + \tau_k\int_0^1 (z_k+y_\infty)_x^2(z_k+y_\infty)_{xx}dx \\
&\leq \tau_k\int_0^1 e^{z_k/2}(z_k+y_\infty)_{xx}^2 dx + \tau_k\int_0^1 e^{3z_k/2}(y_{\infty,x}^2 - y_{\infty,xx})^2 dx \\
&\quad - \tau_k\int_0^1 (z_k+y_\infty)_{xx}^2 dx + \frac{\tau_k}{3}(\beta^3 - \alpha^3).
\end{aligned}$$

In view of (3.3) and (3.4), the right-hand side is uniformly bounded. The left-hand side can be estimated by

$$\int_0^1 (e^{z_k} - e^{z_{k-1}})(1-e^{-z_k})dx \geq \int_0^1 (e^{z_k} - z_k)dx - \int_0^1 (e^{z_{k-1}} - z_{k-1})dx,$$

which is a consequence of the elementary inequality $e^x - 1 \geq x$ for all $x \in \mathbb{R}$. Thus, (3.8) is proved. ∎

Proof of Proposition 3.1. The existence of a solution to (3.1) is shown by the Leray–Schauder fixed-point theorem. For this, let $k \in \{1,\ldots,N\}$ and z_{k-1} be given. Furthermore, let $w \in H^1(0,1)$ and $\sigma \in [0,1]$, and define the linear forms

$$\begin{aligned}
a(z,\phi) &= \int_0^1 e^{w+y_\infty} z_{xx}\phi_{xx}dx, \\
F(\phi) &= -\frac{1}{\tau_k}\int_0^1 e^{y_\infty}(e^w - e^{z_{k-1}})\phi dx - \int_0^1 e^{w+y_\infty} y_{\infty,xx}\phi_{xx}dx,
\end{aligned}$$

where $\phi \in H_0^2(0,1)$. Consider the linear problem

$$a(z,\phi) = \sigma F(\phi) \quad \text{for all } \phi \in H_0^2(0,1).$$

By the Lax–Milgram lemma, there exists a unique solution $z \in H_0^2(0,1)$ to this problem. This defines the fixed-point operator $S : H^1(0,1)\times[0,1] \to H^1(0,1)$, $S(w,\sigma) = z$. It is not difficult to show that S is continuous and compact, since the embedding $H_0^2(0,1) \hookrightarrow H^1(0,1)$ is compact. Moreover, $S(w,0) = 0$ for all $w \in H^1(0,1)$. It remains to prove that any fixed point of S satisfies a uniform bound in $H^1(0,1)$. In fact, Lemma 3.3 shows that any fixed point $z \in H_0^2(0,1)$ is uniformly bounded if $\sigma = 1$. The estimate for $\sigma < 1$ is similar (and, in fact, independent of σ). This provides the wanted bound in $H^1(0,1)$, and the Leray–Schauder theorem can be applied to yield the existence of a solution to (3.1). $\square$

For the proof of Theorem 1.2 we need some more uniform estimates. Let $z^{(N)}$ be defined by $z^{(N)}(x,t) = z_k(x)$ if $t \in (t_{k-1}, t_k]$, $x \in (0,1)$.

LEMMA 3.4. *The following estimates hold:*

$$\|z^{(N)}\|_{L^\infty(0,T;L^1(0,1))} + \|z^{(N)}\|_{L^2(0,T;H^2(0,1))} \le c, \tag{3.9}$$

$$\|z^{(N)}\|_{L^{5/2}(0,T;W^{1,\infty}(0,1))} + \|e^{z^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(0,1))} \le c, \tag{3.10}$$

where $c > 0$ depends only on u_I and the boundary data.

Proof. The inequality $e^x - x \ge |x|$ for all $x \in \mathbb{R}$ and the estimate (3.8) imply that $z^{(N)}$ is uniformly bounded in $L^\infty(0,T;L^1(0,1))$ which, together with (3.8), shows (3.9). Then, using the Poincaré and Gagliardo–Nirenberg inequalities, we obtain from (3.8)

$$\begin{aligned}\|z^{(N)}\|_{L^{5/2}(0,T;W^{1,\infty}(0,1))} &\le c\|z_x^{(N)}\|_{L^{5/2}(0,T;L^\infty(0,1))} \\ &\le c\|z^{(N)}\|_{L^\infty(0,T;L^1(0,1))}^{1/5}\|z^{(N)}\|_{L^2(0,T;H^2(0,1))}^{4/5} \le c.\end{aligned}$$

This estimate, (3.2), and the first bound in (3.9) imply (3.10) since

$$\begin{aligned}\|e^{z^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(0,1))} &\le c\left(\|e^{z^{(N)}}\|_{L^{5/2}(0,T;L^1(0,1))} + \|(e^{z^{(N)}})_x\|_{L^{5/2}(0,T;L^1(0,1))}\right) \\ &\le c\|e^{z^{(N)}}\|_{L^{5/2}(0,T;L^1(0,1))} \\ &\quad + c\|e^{z^{(N)}}\|_{L^\infty(0,T;L^1(0,1))}\|z_x^{(N)}\|_{L^{5/2}(0,T;L^\infty(0,1))} \\ &\le c.\end{aligned}$$

The lemma is proved. $\square$

We also need an estimate for the discrete time derivative. For this, introduce the shift operator $(\sigma_N(z^{(N)}))(\cdot,t) = z_{k-1}$ for $t \in (t_{k-1}, t_k]$.

LEMMA 3.5. *The following estimate holds:*

$$\|e^{z^{(N)}} - e^{\sigma_N(z^{(N)})}\|_{L^{10/9}(0,T;H^{-2}(0,1))} \le c\tau, \tag{3.11}$$

where $c > 0$ depends only on u_I and u_∞.

Proof. From (3.1) and Hölder's inequality we obtain

$$\begin{aligned}\frac{1}{\tau_k}\|e^{z^{(N)}} - e^{\sigma_N(z^{(N)})}\|_{L^{10/9}(0,T;H^{-2}(0,1))} &\le \|e^{z^{(N)}+y_\infty}(z^{(N)} + y_\infty)_{xx}\|_{L^{10/9}(0,T;L^2(0,1))} \\ &\le \|e^{z^{(N)}+y_\infty}\|_{L^{5/2}(0,T;L^\infty(0,1))}\|(z^{(N)} + y_\infty)_{xx}\|_{L^2(0,T;L^2(0,1))}\end{aligned}$$

and the right-hand side is uniformly bounded by (3.9) and (3.10) since $W^{1,1}(0,1) \hookrightarrow L^\infty(0,1)$. $\square$

Proof of Theorem 1.2. For any $N \in \mathbb{N}$, there exists a solution $z^{(N)} \in L^2(0,T; H_0^2(0,1))$ to the sequence of recursive equations (3.1) satisfying $z^{(N)}(\cdot,0) = z_0$. The uniform bounds (3.10) and (3.11) and the compact embedding $W^{1,1}(0,1) \hookrightarrow L^1(0,1)$ allow us to apply Theorem 5 of [17] (Aubin's lemma) yielding the existence of a subsequence of $e^{z^{(N)}}$ (not relabeled) such that $e^{z^{(N)}} \to \rho$ strongly in $L^1(0,T;L^1(0,1))$ and hence also in $L^1(0,T;H^{-2}(0,1))$. The above results give, using (3.2) and $L^1(0,1) \hookrightarrow H^{-2}(0,1)$,

$$\begin{aligned}
\|e^{z^{(N)}} - \rho\|^2_{L^2(0,T;H^{-2}(0,1))} &\le \|e^{z^{(N)}} - \rho\|_{L^\infty(0,T;H^{-2}(0,1))} \|e^{z^{(N)}} - \rho\|_{L^1(0,T;H^{-2}(0,1))} \\
&\le c\left(\|e^{z^{(N)}}\|_{L^\infty(0,T;L^1(0,1))} + \|\rho\|_{L^\infty(0,T;L^1(0,1))}\right) \\
&\quad \times \|e^{z^{(N)}} - \rho\|_{L^1(0,T;H^{-2}(0,1))} \\
&\le c\|e^{z^{(N)}} - \rho\|_{L^1(0,T;H^{-2}(0,1))} \to 0 \quad \text{as } N \to \infty. \qquad (3.12)
\end{aligned}$$

Moreover, the estimate (3.9) provides the existence of a subsequence, also denoted by $z^{(N)}$, such that

$$z^{(N)} \rightharpoonup z \quad \text{weakly in } L^2(0,T;H^2(0,1)) \text{ as } N \to \infty. \qquad (3.13)$$

We claim now that $e^z = \rho$. For this, let w be a smooth function. Letting $N \to \infty$ in

$$0 \le \int_0^T \langle e^{z^{(N)}} - e^w, z^{(N)} - w\rangle_{H^{-2},H_0^2} dt$$

and using the convergence results (3.12) and (3.13) yield

$$0 \le \int_0^T \int_0^1 (\rho - e^w)(w - z)dxdt.$$

The strict monotonicity of $x \mapsto e^x$ then implies that $e^z = \rho$.

Thus, $e^{z^{(N)}} \to e^z$ strongly in $L^1(0,T;L^1(0,1))$ and (maybe for a subsequence) a.e. The uniform bound (3.10) implies that (after extracting a subsequence) $e^{z^{(N)}} \rightharpoonup e^z$ weakly* in $L^{5/2}(0,T;L^\infty(0,1))$ since $W^{1,1}(0,1) \hookrightarrow L^\infty(0,1)$. Therefore, we conclude via Lebesgue's convergence theorem that

$$e^{z^{(N)}} \to e^z \quad \text{strongly in } L^2(0,T;L^2(0,1)). \qquad (3.14)$$

Finally, the uniform estimate (3.11) gives for a subsequence

$$\frac{1}{\tau}(e^{z^{(N)}} - e^{\sigma_N(z^{(N)})}) \rightharpoonup (e^z)_t \quad \text{weakly in } L^{10/9}(0,T;H^{-2}(0,1)). \qquad (3.15)$$

The convergence results (3.13)–(3.15) allow us to pass to the limit $N \to \infty$ in the weak formulation of (3.1) to obtain a weak solution $z \in L^2(0,T;H_0^2(0,1))$ to

$$e^{y_\infty}(e^z)_t = -(e^{z+y_\infty}(z+y_\infty)_{xx})_{xx} \quad \text{in } (0,1),\ t > 0,$$

such that $z(\cdot,0) = z_0 = \log(u_I/u_\infty)$ in the sense of $H^{-2}(0,1)$. Transforming back to the variable $u = e^{z+y_\infty}$ gives the assertion. $\square$

4. Long-time behavior of the solutions. This section is devoted to the proof of Theorem 1.3. The proof is based on the entropy–entropy production method. For this we need the following lemma for lower and upper estimates for the entropy

$$E_3 = \int_0^1 e^{y_\infty}(e^z(z-1)+1)dx.$$

LEMMA 4.1. *Let z, $y_\infty \in L^\infty(0,1)$. Then*

$$c_1\left(\int_0^1 e^{y_\infty}|e^z-1|dx\right)^2 \le E_3 \le c_2\|e^{z/2}-1\|^2_{L^\infty(0,1)}, \tag{4.1}$$

where c_1, $c_2 > 0$ depend on $\|e^{y_\infty}\|_{L^\infty(0,1)}$ and $\|e^z\|_{L^1(0,1)}$.

The lower bound for E_3 is a Csiszar–Kullback-type inequality. A similar version of this lemma is shown in [15].

Proof. The upper bound is proved by expanding the function $f(x) = x^2(\log x^2 - 1)+1$ around $x=1$,

$$\begin{aligned} f(e^{z/2}) &= f(1) + f'(1)(e^{z/2}-1) + \frac{1}{2}f''(\theta)(e^{z/2}-1)^2 \\ &= 2(\log\theta+1)(e^{z/2}-1)^2 \le 2(e^{z/2}+1)(e^{z/2}-1)^2, \end{aligned}$$

where θ lies between $e^{z/2}$ and 1, and using the inequality $\log\theta \le \theta-1 \le \max\{e^{z/2},1\}-1 \le e^{z/2}$. Then

$$E_3 \le 2\int_0^1 e^{y_\infty}(e^{z/2}+1)(e^{z/2}-1)^2dx \le 2\|e^{y_\infty}\|_{L^\infty(0,1)}(\|e^z\|^{1/2}_{L^1(0,1)}+1)\|e^{z/2}-1\|^2_{L^\infty(0,1)},$$

and we set $c_2 = 2\|e^{y_\infty}\|_{L^\infty(0,1)}(\|e^z\|^{1/2}_{L^1(0,1)}+1)$.

For the lower bound we observe that a Taylor expansion of $f(x) = x(\log x-1)+1$ around $x=1$ yields

$$e^{2y_\infty}\left(e^z(z-1)+1\right) = \frac{e^{2y_\infty}}{2\theta}(e^z-1)^2,$$

and $\theta = \theta(z)$ lies between e^z and 1. Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \int_0^1 e^{y_\infty}|e^z-1|dx &\le \int_{\{z<0\}} e^{y_\infty}(1-e^z)dx + \int_{\{z>0\}} e^{y_\infty}(e^z-1)dx \\ &\le \int_{\{z<0\}} e^{y_\infty}\frac{1-e^z}{\theta(z)^{1/2}}dx + \int_{\{z>0\}} e^{y_\infty}\frac{e^z-1}{\theta(z)^{1/2}}\theta(z)^{1/2}dx \\ &\le \text{meas}\{z<0\}^{1/2}\left(\int_{\{z<0\}} e^{2y_\infty}\frac{(1-e^z)^2}{\theta(z)}dx\right)^{1/2} \\ &\quad + \left(\int_{\{z>0\}} e^{2y_\infty}\frac{(e^z-1)^2}{\theta(z)}dx\right)^{1/2}\left(\int_{\{z>0\}}\theta(z)dx\right)^{1/2} \\ &\le (1+\|e^z\|^{1/2}_{L^1(0,1)})\left(\int_0^1 e^{2y_\infty}\frac{(e^z-1)^2}{\theta(z)}dx\right)^{1/2} \\ &\le \sqrt{2}\|e^{y_\infty}\|^{1/2}_{L^\infty(0,1)}(1+\|e^z\|^{1/2}_{L^1(0,1)})E_3^{1/2}, \end{aligned}$$

and the assertion follows with $c_1^{-1} = 2\|e^{y_\infty}\|_{L^\infty(0,1)}(1 + \|e^z\|_{L^1(0,1)}^{1/2})^2$. $\square$

Proof of Theorem 1.3. The idea is to differentiate the entropy E_3 of the introduction with respect to time and to use the differential equation (1.1). Since we do not have enough regularity for the solution u to (1.1), we need to regularize. We set as in the proof of Theorem 1.2 $u_\infty = e^{y_\infty}$, where u_∞ is the unique solution to (1.4). There exist numbers a, $b \in \mathbb{R}$ such that $e^{y_\infty} y_{\infty,xx} = ax + b \le 0$ for all $x \in (0,1)$ since $y_\infty = \log u_\infty$ is assumed to be concave. This implies that $y_\infty \ge \min\{y_\infty(0), y_\infty(1)\}$ and hence $e^{y_\infty} \ge \min\{u_0, u_1\}$ in $(0,1)$. Furthermore, let $z_k \in H_0^2(0,1)$ be a solution to (3.1) for given z_{k-1}. We assume for simplicity that $\tau = \tau_k$ for all $k \in \mathbb{N}$.

Using z_k as a test function in the weak formulation of (3.1), we obtain, after integrating by parts,

$$\begin{aligned}
&\frac{1}{\tau}\int_0^1 e^{y_\infty}(e^{z_k} - e^{z_{k-1}})z_k dx = -\int_0^1 e^{z_k + y_\infty}(z_k + y_\infty)_{xx} z_{k,xx} dx \\
&= -\int_0^1 e^{z_k + y_\infty} z_{k,xx}^2 dx - \int_0^1 e^{z_k} z_{k,xx}(ax+b)dx \\
&= -\int_0^1 e^{z_k + y_\infty} z_{k,xx}^2 dx + \int_0^1 e^{z_k} z_{k,x}^2 (ax+b)dx + a\int_0^1 e^{z_k} z_{k,x} dx \qquad (4.2)\\
&\le -\min\{u_0, u_1\}\int_0^1 e^{z_k} z_{k,xx}^2 dx,
\end{aligned}$$

since $ax + b \le 0$ in $(0,1)$ and $e^{z_k(x)} = 1$ for $x = 0, 1$. The left-hand side is estimated from below by employing the elementary inequality $e^x \ge x + 1$ for all $x \in \mathbb{R}$:

$$\begin{aligned}
&\frac{1}{\tau}\int_0^1 e^{y_\infty}(e^{z_k} - e^{z_{k-1}})z_k dx \\
&= \frac{1}{\tau}\int_0^1 e^{z_k + y_\infty}(z_k - 1)dx - \frac{1}{\tau}\int_0^1 e^{z_{k-1} + y_\infty}(z_{k-1} - 1)dx \\
&\quad + \frac{1}{\tau}\int_0^1 e^{z_{k-1} + y_\infty}(e^{z_k - z_{k-1}} + z_{k-1} - z_k - 1)dx \\
&\ge \frac{1}{\tau}\int_0^1 e^{z_k + y_\infty}(z_k - 1)dx - \frac{1}{\tau}\int_0^1 e^{z_{k-1} + y_\infty}(z_{k-1} - 1)dx. \qquad (4.3)
\end{aligned}$$

This shows that the sequence $E^{(k)} = \int_0^1 e^{y_\infty}(e^{z_k}(z_k - 1) + 1)dx$ is nonincreasing and bounded from below by $E^{(0)} = \int_0^1 (u_I(\log(u_I/u_\infty) - 1) + 1)dx$, which is finite.

We relate the entropy production term on the right-hand side of (4.2) to the entropy itself. We first claim that

$$\int_0^1 e^{z_k} z_{k,xx}^2 dx \ge 4\int_0^1 (e^{z_k/2})_{xx}^2 dx. \qquad (4.4)$$

To see this we set $u = e^{z_k}$ and observe that an integration by parts yields

$$\int_0^1 \frac{u_{xx}u_x^2}{u^2} dx = \frac{2}{3}\int_0^1 \frac{u_x^4}{u^3} dx.$$

Then

$$\int_0^1 e^{z_k} z_{k,xx}^2 dx = \int_0^1 \left(\frac{u_{xx}^2}{u} - \frac{1}{3}\frac{u_x^4}{u^3} \right) dx \geq \int_0^1 \left(\frac{u_{xx}^2}{u} - \frac{5}{12}\frac{u_x^4}{u^3} \right) dx \tag{4.5}$$
$$= 4\int_0^1 (\sqrt{u})_{xx}^2 dx = 4\int_0^1 (e^{z_k/2})_{xx}^2 dx.$$

We need the Poincaré inequalities

$$\|u\|_{L^2(0,1)} \leq \frac{1}{\pi}\|u_x\|_{L^2(0,1)}, \quad \|u\|_{L^\infty(0,1)} \leq \|u_x\|_{L^2(0,1)}$$

for all $u \in H_0^1(0,1)$. Therefore, using Lemma 4.1, we infer

$$\int_0^1 e^{z_k} z_{k,xx}^2 dx \geq 4\pi^2 \int_0^1 (e^{z_k/2} - 1)_x^2 dx \geq 4\pi^2 \|e^{z_k/2} - 1\|_{L^\infty(0,1)}^2 \geq \frac{4\pi^2}{c_2} E^{(k)}. \tag{4.6}$$

Setting $\gamma = 4\pi^2 \min\{u_0, u_1\}/c_2$, we obtain from (4.2) the difference inequality

$$E^{(k)} \leq E^{(k-1)} - \gamma\tau E^{(k)},$$

from which

$$E^{(k)} \leq (1+\gamma\tau)^{-1} E^{(k-1)} \leq (1+\gamma\tau)^{-k} E^{(0)} \leq (1+\gamma\tau)^{-t/\tau} E^{(0)} \tag{4.7}$$

follows. The parameter γ depends on $\|e^{z_k}\|_{L^1(0,1)}$ through c_2. However, since $e^{z^{(N)}}$ is uniformly bounded in $L^\infty(0,T;L^1(0,1))$ in view of Lemma 3.3, γ is bounded uniformly in k. We have shown in the proof of Theorem 1.2 that $e^{z_k} \to e^z$ a.e. Then the uniform boundedness of e^{z_k} and z_k and Lebesgue's dominated convergence theorem imply that

$$E^{(k)} \to E_3(t) = \int_0^1 e^{y_\infty} (e^{z(\cdot,t)}(z(\cdot,t) - 1) + 1) dx.$$

Hence, after letting $\tau \to 0$, we conclude from (4.7) that $E_3(t) \leq E_3(0)e^{-\gamma t}$. The first inequality in (4.1) gives the assertion with $\lambda = \gamma/2$. $\square$

Remark 4.2. The decay rate λ is not optimal. For instance, we neglected the term $\int_0^1 u_x^4/12u^3 dx$ in (4.5) and the constants in (4.1) are not the best ones. For optimal constants in logarithmic Sobolev inequalities related to (1.1) with periodic boundary conditions, we refer the reader to [10].

Remark 4.3. It is not easy to find conditions on the boundary data for which $\log u_\infty$ is concave. An example is $u_0 = u_1$ and $w_0 = -w_1 \geq 0$. Indeed, if $y = \log u_\infty$, we have $y(0) = y(1)$ and $y_x(0) = -y_x(1) \geq 0$ and therefore, y is symmetric around $x = \frac{1}{2}$. Thus (see Remark 2.3) $a = e^{y_0}(y_{xx}(1) - y_{xx}(0)) = 0$ and $b = e^{y_0} y_{xx}(0) \leq 0$. This implies $(\log u_\infty)_{xx} = y_{xx} = be^{-y} \leq 0$ in $(0,1)$.

Remark 4.4. The assumption on the concavity of $\log u_\infty$ can be slightly relaxed. Indeed, we claim that the assertion of Theorem 1.3 also holds if $((\log u_\infty)_{xx})^+$ is small enough in the sense

$$4\frac{\max\{u_\infty(x) : 0 \leq x \leq 1\}}{\min\{u_\infty(x) : 0 \leq x \leq 1\}} \int_0^1 ((\log u_\infty)_{xx})^+ dx \leq 1 - \delta \tag{4.8}$$

for some $\delta > 0$, where $(x)^+ = \max\{0, x\}$. We prove this result by deriving a bound on the second integral in (4.2) in terms of the first one, employing the weighted Poincaré inequality [7, Thm. 1.4]

$$\int_0^1 u_x^2 \mu(x) dx \le K \int_0^1 u_{xx}^2 dx$$

for all $u \in H^2(0,1)$ satisfying $u(0) = u(1)$ (which implies that $\int_0^1 u_x dx = 0$). The function μ is assumed to be nonnegative and measurable. The best constant $K > 0$ is not explicit but can be bounded by $K \le 4\int_0^1 \mu(x)dx$ [7, Rem. 1.10.4]. We choose $\mu(x) = (ax+b)^+ = (u_\infty(\log u_\infty)_{xx})^+$. Then the weighted Poincaré inequality and (4.4) give

$$\begin{aligned}\int_0^1 e^{z_k + y_\infty} z_{k,xx}^2 dx &\ge 4m \int_0^1 (e^{z_k/2})_{xx}^2 dx \ge \frac{4m}{K} \int_0^1 (e^{z_k/2})_x^2 \mu(x) dx \\ &= \frac{m}{K} \int_0^1 (ax+b)^+ e^{z_k} z_{k,x}^2 dx,\end{aligned}$$

where $m = \min\{u_\infty(x) : 0 \le x \le 1\}$. Inserting this inequality into (4.2) and using (4.3), we obtain

$$\begin{aligned}\frac{1}{\tau}\left(E^{(k)} - E^{(k-1)}\right) &\le -\int_0^1 e^{z_k + y_\infty} z_{k,xx}^2 dx + \int_0^1 (ax+b)^+ e^{z_k} z_{k,x}^2 dx \\ &\le \left(\frac{K}{m} - 1\right) \int_0^1 e^{z_k + y_\infty} z_{k,xx}^2 dx.\end{aligned}$$

Assumption (4.8) shows that $K/m \le 1 - \delta$ and hence, by (4.6),

$$\frac{1}{\tau}\left(E^{(k)} - E^{(k-1)}\right) \le -\delta \int_0^1 e^{z_k + y_\infty} z_{k,xx}^2 dx \le -\frac{4\pi^2 \delta m}{c_2} E^{(k)}.$$

Now proceed as in the proof of Theorem 1.3. The convergence rate in the L^1 norm is given by $\lambda = 2\pi^2 \delta m / c_2$.

5. Numerical examples. In this section we show by numerical examples that the assumption of concavity of $\log u_\infty$ (or the assumption (4.8)), where u_∞ is the solution to (1.4), seems to be only technical. Equation (1.1) is solved numerically in the formulation

$$u_t = -u_{xxxx} + \left(\frac{u_x^2}{u}\right)_{xx} \quad \text{in } (0,1). \tag{5.1}$$

We use a uniform grid $(x_i, t_j) = (\triangle x \cdot i, \triangle t \cdot j)$ with spatial mesh size $\triangle x = 10^{-3}$ and time step $\triangle t = 10^{-6}$. With the approximation u_{ij} of $u(x_i, t_j)$, the fully implicit discretization reads as

$$\frac{1}{\triangle t}(u_{ij} - u_{i,j-1}) = -D^+D^-D^+D^- u_{ij} + D^+D^-\left(\frac{(D^+u_{ij})^2}{u_{ij}}\right),$$

where D^+ and D^- are the forward and backward difference operators on the spatial mesh (see [13]). The nonlinear equations are solved on each time level by Newton's method where the initial guess is chosen to be the solution of the previous time level.

For the first example we use the boundary conditions

$$u(0,t) = u_0, \quad u(1,t) = u_1, \tag{5.2}$$

$$u_x(0,t) = w_0 = 2\sqrt{u_0}(\sqrt{u_1} - \sqrt{u_0}), \quad u_x(1,t) = w_1 = 2\sqrt{u_1}(\sqrt{u_1} - \sqrt{u_0}), \tag{5.3}$$

with $u_0 \le u_1$. The advantage of these conditions is that the stationary problem (1.4) has the exact solution

$$u_\infty(x) = ((\sqrt{u_1} - \sqrt{u_0})x + \sqrt{u_0})^2, \quad x \in (0,1).$$

We choose the initial condition $u_I(x) = e^{-x}\sin(3\pi x) + 3x + 1$ and the boundary values $u_0 = 1$ and $u_1 = 4$. The numerical solution at various times is displayed in Figure 5.1. The discrete solution seems to converge to the exact solution u_∞ as $t \to \infty$. Figure 5.2 shows the exponential decay of the relative entropy

$$E_3(t) = \int_0^1 u(\cdot,t)((\log(u(\cdot,t)/u_\infty) - 1) + u_\infty)dx$$

and of the L^1 deviation $\|u(\cdot,t) - u_\infty\|_{L^1(0,1)}$. As predicted by the proof of Theorem 1.3, the decay rate of the L^1 deviation is half of the rate of the relative entropy. Notice that the function $\log u_\infty$ is concave; i.e., the assumptions of Theorem 1.3 are satisfied.

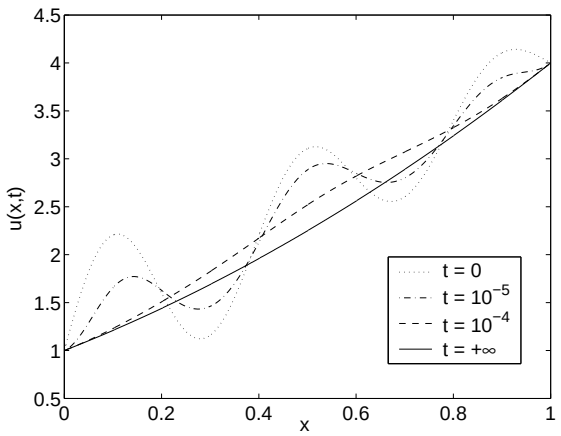

FIG. 5.1. *Numerical solution to* (5.1)–(5.3) *with* $u_0 = 1$, $u_1 = 4$, $w_0 = 2$, *and* $w_1 = 4$ *at various times.*

In the second example we show by a numerical example that the solution to (1.1) decays exponentially fast even if the function $\log u_\infty$ is convex. For this we choose the boundary conditions $u_0 = 1.5$, $u_1 = 0.8$, $w_0 = -4.6127$, and $w_1 = 2.0618$. The stationary solution u_∞ is computed numerically from the equation

$$u_\infty(\log u_\infty)_{xx} = ax + b, \quad x \in (0,1),$$

where $a = 1$ and $b = 3$. Then, $\log u_\infty$ is strictly convex in $(0,1)$ and the assumption (4.8) is not satisfied. We choose the initial function $u_I(x) = -e^{-x}\sin(2\pi x) - \frac{7}{10}x + \frac{3}{2}$. Figure 5.3 shows the discrete solution for various times. In this case, the relative entropy and the L^1 deviation are also exponentially decaying (Figure 5.4) although the condition of Theorem 1.3 is not satisfied. This suggests that the concavity hypothesis is purely technical.

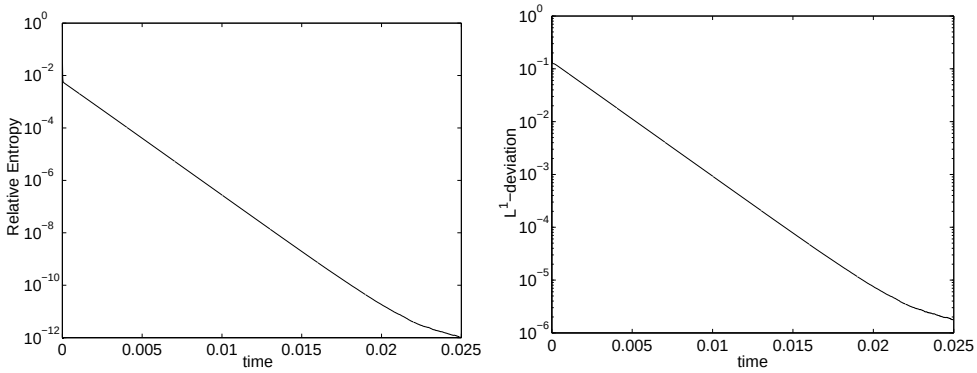

FIG. 5.2. *Logarithmic plot of the relative entropy* $E_3(t)$ *(left) and the* L^1 *deviation* $\|u(\cdot,t) - u_\infty\|_{L^1(0,1)}$ *(right) for the solution to* (5.1)–(5.3) *with* $u_0 = 1$, $u_1 = 4$.

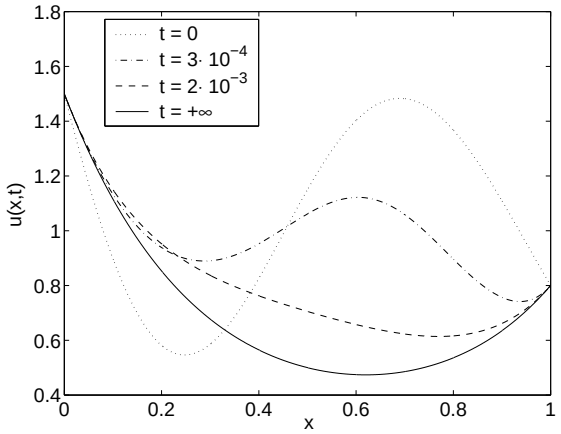

FIG. 5.3. *Numerical solution to* (5.1), (1.3) *with* $u_0 = 1.5$, $u_1 = 0.8$, $w_0 = -4.6127$, *and* $w_1 = 2.0618$ *at various times.*

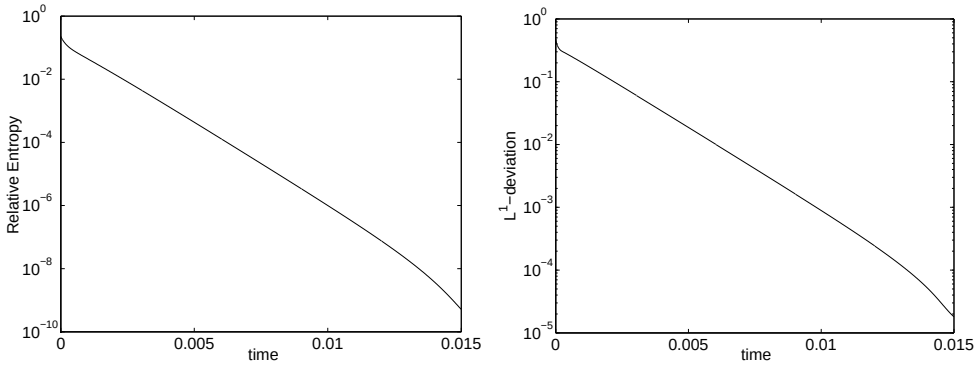

FIG. 5.4. *Logarithmic plot of the relative entropy* $E_3(t)$ *(left) and the* L^1 *deviation* $\|u(\cdot,t) - u_\infty\|_{L^1(0,1)}$ *(right) for the solution to* (5.1), (1.3) *with* $u_0 = 1.5$, $u_1 = 0.8$, $w_0 = -4.6127$, *and* $w_1 = 2.0618$.

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