ESTIMATES FOR RADIAL SOLUTIONS OF THE HOMOGENEOUS LANDAU EQUATION WITH COULOMB POTENTIAL

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Abstract. Motivated by the question of existence of global solutions, we obtain pointwise upper bounds for radially symmetric and monotone solutions to the homogeneous Landau equation with Coulomb potential. The estimates say that blow up in the L^∞ norm at some finite time T occurs only if a certain quotient involving f and its Newtonian potential concentrates near zero, which implies blow up in more standard norms, such as the $L^{3/2}$ norm. This quotient is shown to be always less than a universal constant, suggesting that the problem of regularity for the Landau equation is in some sense critical.

The bounds are obtained using the comparison principle both for the Landau equation and for the associated mass function. In particular, the method provides long-time existence results for a modified version of the Landau equation with Coulomb potential, recently introduced by Krieger and Strain.

1. Introduction

This manuscript is concerned with the Cauchy problem for the homogeneous Landau equation: such equation takes the general form

$$\partial_t f(v,t) = Q(f,f), \quad f(v,0) = f_{\rm in}(v), \quad v \in \mathbb{R}^3, \quad t > 0, \tag{1.1}$$

where $Q(f,f)$ is a quadratic operator known as the Landau collisional operator

$$Q(f,f) = \mathrm{div}\left(\int_{\mathbb{R}^3} A(v-y)\,(f(y)\nabla_v f(v) - f(v)\nabla_y f(y))\,dy\right). \tag{1.2}$$

The term $A(v)$ denotes a positive and symmetric matrix

$$A(v) := C_\gamma \left(\mathrm{Id} - \frac{v\otimes v}{|v|^2}\right)\varphi(|v|), \quad v \neq 0, \quad C_\gamma > 0,$$

which acts as the projection operator onto the space orthogonal to the vector v. The function $\varphi(|v|)$ is a scalar valued function determined from the original Boltzmann kernel describing how particles interact. If the interaction strength between particles at a distance r is proportional to r^{1-s}, then

$$\varphi(|v|) := |v|^{\gamma+2}, \quad \gamma = \frac{(s-5)}{(s-1)}. \tag{1.3}$$

Note that $s = 2$ corresponds to the Coulomb potential, in which case $\gamma = -3$ [18, Chapter 1, Section 1.4]. Any solution to (1.1)-(1.2) is an integrable and nonnegative scalar field $f(v,t) : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^+$. Equation (1.1) describes the evolution of a plasma in spatially homogeneous regimes, which means that the density function f depends only on the velocity component v. Landau's original intent in deriving this approximation was to make sense of the

Boltzmann collisional operator, which always diverges when considering purely grazing collisions.

The Cauchy problem for (1.1)-(1.3) is very well understood for the case of hard potentials, which correspond to $\gamma \geq 0$ above. Desvillettes and Villani showed the existence of global classical solutions for hard potentials and studied its long time behavior, see [3, 4, 18] and references therein. In this case there is a unique global smooth solution, which converges exponentially to an equilibrium distribution, known as the Maxwellian function

$$\mathcal{M}(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}.$$

Analyzing the soft potentials case, $\gamma < 0$, has proved to be more difficult: using a probabilistic approach, the authors in [19, 5, 1] show uniqueness and existence of weak solutions for $\gamma \in [-2, 0]$. For $\gamma \in [-3, -2]$ it is known (i) existence for small time or (ii) global in time existence with smallness assumption on initial data [1, 2]. Finally, for the Coulomb case $\gamma = -3$, Fournier [6] showed the uniqueness of weak solutions as long as they remain in L^∞.

Villani [17] introduced the so called H-solutions, which enjoy (weak) a priori bounds in a weighted Sobolev space. However, the issue of their uniqueness and regularity (i.e. no finite time break down occurs) has remained open, even for smooth initial data: see [18, Chapter 1, Chapter 5] for further discussion.

Guo in [10] employs a completely different approach based on perturbation theory for the existence of periodic solutions to the spatially inhomogeneous Landau equation in $\mathbb{R}^3$. He shows that if the initial data is sufficiently close to the unique equilibrium in a certain high Sobolev norm then a unique global solution exists. Moreover, as remarked in [10], this approach also extends to the case of potentials (1.3) where γ might even take values below -3.

Due to the lack of a global well-posedness theory, several conjectures about possible finite-time blow up for general initial data have been made throughout the years. In [18] Villani discussed the possibility that (1.1)-(1.3) could blow up for $\gamma = -3$. Note that for smooth solutions (1.1)-(1.3) with $\gamma = -3$ can be rewritten as

$$\partial_t f = \text{div}(A[f]\nabla f - f\nabla a[f]) = \text{Tr}(A[f]D^2 f) + f^2, \tag{1.4}$$

where

$$A[f] := A(v) * f = \frac{1}{8\pi|v|}\left(\text{Id} - \frac{v \otimes v}{|v|^2}\right) * f, \quad \Delta a = -f.$$

Equation (1.4) can be thought of as a quasi-linear nonlocal heat equation. Supports for blow-up conjectures were given by the fact that (1.4) is reminiscent of the well studied semilinear heat equation

$$\partial_t f = \Delta f + f^2. \tag{1.5}$$

Blow up for (1.5) is known to happen for every L^p norm, $p > 3/2$, see [7].

However, despite the apparent similarities, equation (1.4) behaves differently from (1.5). The Landau equation admits a richer class of equilibrium solution: every Maxwellian $\mathcal{M}$ solves $Q(\mathcal{M}, \mathcal{M}) = 0$ which holds, in particular, for those with arbitrarily large mass.

From a different perspective, Krieger-Strain [11] considered a modified version of (1.4)

$$\partial_t f = a[f]\Delta f + \alpha f^2, \tag{1.6}$$

and showed global existence of smooth radial solutions starting from radial initial data when $\alpha < 2/3$. This range for α later was expanded to any $\alpha < 74/75$ by means of a non-local inequality obtained by Gressman, Krieger and Strain [8]. Note that when $\alpha = 1$, the above equation can be written in divergence form,

$$\partial_t f = \mathrm{div}(a[f]\nabla f - f\nabla a[f]). \tag{1.7}$$

These results put in evidence how a non-linear equation with a non-local diffusivity such as (1.7) behaves drastically different from (and better than) (1.5).

Our main results in this manuscript are twofold. The first one gives necessary conditions for the finite time blow up of solutions to (1.4). The second (unconditional) result says that solutions to (1.7) do not blow up at all, and in fact become instantaneously smooth (even for initial data that might be initially unbounded). Both results deal only with radially symmetric, decreasing initial conditions; more precisely we assume that

$$\begin{cases} f_{\text{in}} \geq 0, \;\; \int_{\mathbb{R}^3} f_{\text{in}}\, dv = 1, \;\; \int_{\mathbb{R}^3} f_{\text{in}}|v|^2\, dv = 3, \;\; f_{\text{in}} \in L^\infty(\mathbb{R}^3), \\ \int_{\mathbb{R}^3} f_{\text{in}} \log(f_{\text{in}})\, dv < \infty, \quad \text{and} \quad |v| \leq |w| \Rightarrow f_{\text{in}}(v) \geq f_{\text{in}}(w). \end{cases} \tag{1.8}$$

The normalization of the initial data is standard and follows a standard change of variables. The main results are the following.

Theorem 1.1. *Let f_{in} be as in (1.8). Then there exists $T_0 > 0$ and $f : \mathbb{R}^3 \times (0, T_0) \to \mathbb{R}_+$ such that f is smooth and solves (1.4) for $t \in (0, T_0)$, with $f(\cdot, 0) = f_{\text{in}}$. Moreover, T_0 is maximal in the sense that either $T_0 = \infty$ or else the $L^{3/2}$ norm of f accumulates near $v = 0$ as $t \to T_0^-$, in particular*

$$\lim_{t \to T_0^-} \|f(\cdot, t)\|_{L^p(B_1)} = \infty, \;\; \forall\, p > 3/2.$$

In fact, the above theorem is a consequence of the following sharper result.

Theorem 1.2. *There is a constant ε_0, with $\varepsilon_0 \geq 1/96$, such that if above $T_0 < \infty$ then*

$$\limsup_{r \to 0^+} \sup_{t \in (0, T_0)} \left\{ r^2 \frac{\int_{B_r} f(v,t)\, dv}{\int_{B_r} a[f](v,t)\, dv} \right\} \geq \varepsilon_0.$$

Neither of the above theorems are enough to guarantee long time existence of classical solutions to (1.4). However, Theorem 1.2 suggests that (1.4) is in some sense "critical" for regularity. It can be shown (see Proposition 5.6) that for any nonnegative $f \in L^1(\mathbb{R}^3)$

$$r^2 \frac{\int_{B_r} f(v)\, dv}{\int_{B_r} a[f](v)\, dv} \leq 3, \;\; \forall\, r > 0.$$

In particular, if the ε_0 in Theorem 1.2 could be shown to be at least 3 (or in general if the upper bound in the last inequality could be improved to something less than ε_0) it would immediately follow that solutions to the Landau equation (1.4) cannot blow up in finite time. It is not clear if this can be guaranteed for general f without at least using some partial time regularization.

On the other hand, methods used in the proof of Theorem 1.1 and Theorem 1.2 yield long time existence for the modified Landau equation (1.7) (again, in the radial case).

Theorem 1.3. *Let f_{in} be as in (1.8) and such that for some $p > 6$,*

$$f_{\text{in}} \in L^p_{weak}(\mathbb{R}^3).$$

Then, there exists a function $f : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$*, smooth for positive times, with* $f(\cdot, 0) = f_{\text{in}}$ *which solves, for* $t > 0$*,*

$$\partial_t f = a[f]\Delta f + f^2.$$

We approach the analysis from the point of view of nonlinear parabolic equations. The nonlocal dependence of the coefficients on the solution prevents the equation from satisfying a comparison principle: if v_0 is a contact point of two functions f and g, i.e. $f(v_0) = g(v_0)$ and everywhere else $f(v) \leq g(v)$, it does not follow that $Q(f, f)(v_0) \leq Q(g, g)(v_0)$. More precisely, for the case where $Q(f, f)$ corresponds to (1.2) one cannot expect an inequality such as

$$\text{Tr}(A[f]D^2 f)(v_0) \leq \text{Tr}(A[g]D^2 g)(v_0).$$

In fact due to the nonlocality of A one only has $A[f](v_0) \leq A[g](v_0)$. Equality $A[f](v_0) = A[g](v_0)$ holds only when $f \equiv g$ for every $v \in \mathbb{R}^3$. The maximum principle is not useful either, since at a maximum point for f we only obtain $\partial_t f \leq -f\Delta a[f]$, which does not rule out growth of the maximum of f. The same observations apply to $Q(f, f)$ corresponding to (1.7).

On the other hand, if one could construct (using only properties of f that are independent of t) a function $U(v)$ such that

$$\text{Tr}(A[f])D^2 U) + fU \leq 0 \text{ in } \mathbb{R}^3$$
$$(\text{ respectively, } a[f]\Delta U + fU \leq 0 \text{ in } \mathbb{R}^3 \text{ }),$$

then the comparison principle (for linear parabolic equations) would guarantee that $f \leq cU$ for all times provided $f(t = 0) \leq cU$. Our main observation is that (under radial symmetry) the above can be made to work with $U(v) = |v|^{-\gamma}$, $\gamma \in (0, 1)$. From here higher local integrability of f can be propagated, and from there higher regularity follows by standard elliptic regularization.

A previous attempt by the authors, also based on upper barrier arguments (but meant to cover any bounded, fast decaying initial data), was ultimately undone by a computational error. However, Theorems 1.1-1.3 show that the use of upper barriers to study (1.4) is fruitful at least for radially symmetric and decreasing initial conditions. On the other hand, the authors in [9] have shown a local L^∞-regularization estimate using De Giorgi iteration method for $\gamma > -2$.

Remark 1.4. After the submission of this article, the authors have learned of related work of Silvestre [16] on the Boltzmann equation, covering the spatially inhomogeneous. In [16] a priori estimates rely on maximum principle arguments and make use of the regularity for parabolic integro-differential equations, particularly recent work of Schwab and Silvestre [15].

1.1. **Outline.** The rest of the paper is organized as follows. After a brief review in Section 2 on nonlinear parabolic theory that will be needed to construct local solutions to the non-linear problems, in Section 3 we outline the symmetry properties of (1.4). Section 4 deals with short time existence. In Section 5 we present a barrier argument that will allow to prove conditional non-blow up results for the Landau equation and global well-posedness for the modified Landau equation in Section 6.

1.2. **Notation.** Universal constants will be denoted by c, c_0, c_1, C_0, C_1, C. Vectors in $\mathbb{R}^3$ will be denoted by v, w, x, y and so on, the inner product between v and w will be written (v, w). $B_R(v_0)$ denotes the closed ball of radius R centered at v_0, if $v_0 = 0$ we simply write B_R. The identity matrix will be noted by $\mathbb{I}$, the trace of a matrix X will be denoted $\text{Tr}(X)$. The initial condition for the Cauchy problem will always be denoted by f_{in}.

The letter Ω denotes a general compact subset of $\mathbb{R}^3$. $Q \subset \mathbb{R}^3 \times \mathbb{R}_+$ is a space-time cylinder of parabolic diameter R with $R > 0$ a general constant, unless otherwise specified. $\partial_p Q$ denotes the parabolic boundary of Q.

1.3. **Acknowledgements.** MPG is supported by NSF DMS-1310746 and DMS-1412748. NG is supported by NSF-DMS 1201413. The authors would like to thank MSRI for the hospitality during the program Free Boundary Problems, Theory and Applications in the Spring of 2011, where this work was started. We also would like to thank Luis Silvestre and Cedric Villani for many fruitful communications. MPG would like to thank NCTS Mathematics Division Taipei for their kind hospitality. Last but not least, the authors would like to express their gratitude to the anonymous referees for their thoughtful and thorough reading of the original manuscript, as well as for their many useful suggestions.

2. A rapid review of linear parabolic equations

We will work with two bilinear operators, namely the one associated to (1.4)

$$Q_{\mathcal{L}}(g, f) := \operatorname{div}(A[g]\nabla f - f\nabla a[g]) = \operatorname{tr}[A[g]D^2 f] + fg,$$

and the one associated to (1.7),

$$Q_{\mathcal{KS}}(g, f) := \operatorname{div}(a[g]\nabla f - f\nabla a[g]) = a[g]\Delta f + fg.$$

As it is well known, through $Q_{\mathcal{L}}$ (and also $Q_{\mathcal{KS}}$) any $g : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$, gives rise to a linear elliptic operator with variable coefficients, as follows:

$$\begin{aligned} \phi \to Q_{\mathcal{L}}(g, \phi) &:= \operatorname{div}(A[g]\nabla \phi - \phi\nabla a[g]) = \operatorname{tr}(A[g]D^2\phi) + \phi g, \\ \phi \to Q_{\mathcal{KS}}(g, \phi) &:= \operatorname{div}(a[g]\nabla \phi - \phi\nabla a[g]) = a[g]\Delta\phi + \phi g. \end{aligned}$$

Accordingly, given such a g and initial data f_{in}, one considers the linear Cauchy problem,

$$\begin{cases} \partial_t f &= Q(g, f) \text{ in } \mathbb{R}^3 \times \mathbb{R}_+, \\ f(\cdot, 0) &= f_{\text{in}}, \end{cases} \tag{2.1}$$

both when $Q = Q_{\mathcal{L}}$ or $Q = Q_{\mathcal{KS}}$.

Remarks 1. Note that $Q_{\mathcal{L}}(g, f)$ and $Q_{\mathcal{KS}}(g, f)$ can both be expressed as a divergence, so any solution to (2.1) preserves its mass over time, i.e.

$$\|f(\cdot, t)\|_{L^1(\mathbb{R}^3)} = \|f_{\text{in}}(\cdot)\|_{L^1(\mathbb{R}^3)} =: M_{in} \text{ for all } t > 0.$$

Lemma 2.1. *See* ([12] Thm 5.1 Page 320) *Let $f_{\text{in}} : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ be non-negative functions such that for some $\beta \in (0, 1)$ we have*

$$\begin{aligned} & f_{\text{in}} \in L^1(\mathbb{R}^3) \cap C^{2+\beta}(\mathbb{R}^3), \\ & A[g], \nabla a[g] \in C^{\beta,\beta/2}(\mathbb{R}^3 \times \mathbb{R}_+). \end{aligned} \tag{2.2}$$

Then, for every $\delta > 0$ there exists a unique $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ with $f \in C^{2+\beta,1+\beta/2}(\mathbb{R}^3 \times \mathbb{R}_+)$ which is a classical solution of

$$\begin{cases} \partial_t f &= \delta\Delta f + Q(g, f) \text{ in } \mathbb{R}^3 \times \mathbb{R}_+, \\ f(\cdot, 0) &= f_{\text{in}}, \end{cases} \tag{2.3}$$

where $Q(\cdot, \cdot)$ denotes either $Q = Q_{\mathcal{L}}$ or $Q = Q_{\mathcal{KS}}$.

Next we summarize in three theorems several classical local regularity estimates for parabolic equations of the form

$$\partial_t f = \mathrm{div}\,(B\nabla f + fb)\,,$$

where $f : Q \to \mathbb{R}$ and $Q = B_R(v_0) \times (t_0 - R^2, t_0) \subset \mathbb{R}^d \times \mathbb{R}_+$ is the parabolic cylinder of radius R centered at some points x_0, t_0. The first two theorems are respectively a local Hölder estimate (from De Giorgi-Nash-Moser theory) and a L^∞ estimate for f in terms of its boundary data (Stampacchia estimate), see [12, Chapter III, Theorem 10.1, page 204] and [12, Chapter IV, Theorem 10.1, page 351] as well as [14, Chapter VI, Theorem 6.29 p. 131] for the respective proofs. The main point of these theorems is that they do not require any regularity assumption on the diffusion matrix B (beyond ellipticity and boundedness).

Theorem 2.2. *(De Giorgi-Nash-Moser estimate.) Suppose f is a weak solution of the equation*

$$\partial_t f = \mathrm{div}\,(B\nabla f + fb)\,,$$

where b is a vector field and B is a symmetric matrix such that

$$\lambda\,\mathbb{I} \leq B(v,t) \leq \Lambda\,\mathbb{I} \text{ a.e. in } Q.$$

Then, there is some $\alpha \in (0,1)$ and $C > 0$ such that the following estimate holds:

$$[f]_{C^{\alpha,\alpha/2}(Q_{1/2})} \leq C\left(\|f\|_{L^\infty(Q)} + R^2\|b\|_{L^\infty(Q)}\right), \tag{2.4}$$

where $Q_{1/2} := B_{R/2}(x_0) \times (t_0 - (R/2)^2, t_0)$ and α and C are determined by λ, Λ, R and d.

Theorem 2.3. *(Stampacchia estimate.) If f is a weak solution of*

$$\partial_t f \leq \mathrm{div}\,(B\nabla f + b)\,,$$

with B and b as in the previous theorem, there exists a constant $C > 0$ such that

$$\|f\|_{L^\infty(Q)} \leq C\left(\|f\|_{L^1(Q)} + \|b\|_{L^\infty(Q)}\right), \tag{2.5}$$

as before, C is determined by λ, Λ, d and R.

The last theorem recalls interior classical regularity estimates when the coefficients are Hölder continuous in time and space. See [12, Chapter IV] or also [14, Chapter III, Theorem 6.17] for a proof.

Theorem 2.4. *(Schauder estimates.) If $B, b \in C^{\beta;\beta/2}(Q)$, then there is a finite C such that*

$$[D^2 f]_{C^{\beta,\beta/2}(Q_{1/2})} + [\partial_t f]_{C^{\beta,\beta/2}(Q_{1/2})} \leq C\left(\lambda, \Lambda, R, \|B\|_{C^{\beta;\beta/2}(Q)}, \|b\|_{C^{\beta;\beta/2}(Q)}, \|f\|_{L^\infty(Q)}\right).$$

3. Radial symmetry

This section is devoted to some technical lemmas. The proofs of the first two propositions are rather technical and can be found in the Appendix.

Proposition 3.1. *Suppose f_{in} and $g(\cdot,t)$ are both radially symmetric, and let $Q(\cdot,\cdot)$ denote either $Q_{\mathcal{L}}$ or $Q_{\mathcal{KS}}$. Then any solution of the linear Cauchy problem*

$$\partial_t f = Q(g,f), \quad f(v,0) = f_{\mathrm{in}}(v),$$

is radially symmetric for all t. Furthermore, if f_{in} and g are radially decreasing, then so is f.

Let $h : \mathbb{R}^3 \to \mathbb{R}_+$, define

$$A^*[h](v) := (A[h](v)\hat{v}, \hat{v}), \quad v \neq 0, \quad \hat{v} := v|v|^{-1}. \tag{3.1}$$

There are two useful expressions for $A^*[h]$ and $a[h]$ when h is radially symmetric.

Proposition 3.2. *Let $h \in L^1(\mathbb{R}^3)$ be radially symmetric and non-negative. Then*

$$A^*[h](v) = \frac{1}{12\pi|v|^3} \int_{B_{|v|}} h(w)|w|^2 \, dw + \frac{1}{12\pi} \int_{B^c_{|v|}} \frac{h(w)}{|w|} \, dw, \tag{3.2}$$

$$a[h](v) = \frac{1}{4\pi|v|} \int_{B_{|v|}} h(w) \, dw + \frac{1}{4\pi} \int_{B^c_{|v|}} \frac{h(w)}{|w|} \, dw. \tag{3.3}$$

The second formula above is simply the classical formula for the Newtonian potential in the case of radial symmetry; the formula for $A^*[h]$ is new and the proof can be found in the Appendix.

Lemma 3.3. *Let $h \in L^1(\mathbb{R}^3)$ be a non-negative, spherically symmetric function*

(1) If h is monotone decreasing with $|v|$, and

$$\int_{B_{R_1} \setminus B_{R_0}} h \, dv \geq \theta > 0,$$

for some $\delta > 0$ and $0 < R_0 < R_1$ then,

$$A[h](v) \geq \frac{\theta R_0^2}{12\pi(1 + R_1^3)} \frac{1}{1 + |v|^3} \mathbb{I}. \tag{3.4}$$

(2) If h is bounded, i.e. if $\|h\|_{L^\infty(\mathbb{R}^3)} = h(0) < +\infty$, then

$$A[h](v) \leq a[h]\mathbb{I} \leq 2 \left(\frac{\|h\|_{L^\infty(\mathbb{R}^3)} + \|h\|_{L^1(\mathbb{R}^3)}}{1 + |v|} \right) \mathbb{I}, \quad \forall\, v \in \mathbb{R}^3. \tag{3.5}$$

Proof. (1) Let $A^*[h]$ be as in (3.2). If $|v| \geq R_1$, then

$$\begin{aligned} A^*[h](v) &\geq \frac{1}{12\pi|v|^3} \int_{B_{R_1}} h(w)|w|^2 \, dw \geq \frac{1}{12\pi|v|^3} \int_{B_{R_1} \setminus B_{R_0}} h(w)|w|^2 \, dw \\ &\geq \frac{R_0^2}{12\pi|v|^3} \int_{B_{R_1} \setminus B_{R_0}} h(w,t) \, dw \geq \frac{\theta R_0^2}{12\pi|v|^3}. \end{aligned}$$

Note that Proposition (3.2) guarantees that $A^*[h]$ is radially decreasing. Thus,

$$A^*[h](v) \geq \frac{\theta R_0^2}{12\pi R_1^3}, \quad \forall\, v \in B_{R_1}.$$

Combining both estimates, we conclude that

$$A^*[h](v) \geq \frac{\theta R_0^2}{12\pi(1 + R_1^3)} \frac{1}{1 + |v|^3}.$$

(2) If $h \in L^\infty$, we may use (3.3) to obtain the estimate

$$\begin{aligned} A[h] \leq a[h](v)\mathbb{I} &\leq \left(\frac{h(0)}{4\pi|v|} \int_{B_{|v|}} dw + \frac{1}{4\pi} \int_{B^c_{|v|}} h(w) \, dw \right) \mathbb{I} \\ &\leq \left(\|h\|_{L^\infty(\mathbb{R}^3)} + \|h\|_{L^1(\mathbb{R}^3)} \right) \mathbb{I}, \quad \text{if } |v| \leq 1, \end{aligned}$$

and

$$A[h] \leq a[h](v)\mathbb{I} \leq \left(\frac{\|h\|_{L^1(\mathbb{R})}}{2\pi|v|}\right)\mathbb{I} \leq \left(\frac{\|h\|_{L^1(\mathbb{R})}}{1+|v|}\right)\mathbb{I} \quad \text{if } |v| \geq 1.$$

□

Proposition 3.4. *Let h be a positive and radially symmetric and decreasing function. For any $\gamma \in (0,1)$ define $U_\gamma(v)$ as*

$$U_\gamma(v) := |v|^{-\gamma}.$$

Then, for $Q = Q_{\mathcal{L}}$ or $Q = Q_{\mathcal{KS}}$

$$Q(h, U_\gamma) \leq U_\gamma \left(-\tfrac{1}{3}\gamma(1-\gamma)a[h]|v|^{-2} + h\right).$$

Proof. As U_γ is radial

$$\nabla U_\gamma(v) = U_\gamma'(v)\tfrac{v}{|v|}, \quad D^2U_\gamma(v) = U_\gamma''(v)\tfrac{v}{|v|} \otimes \tfrac{v}{|v|} + U_\gamma'(v)\tfrac{1}{|v|}(\mathbb{I} - \tfrac{v}{|v|} \otimes \tfrac{v}{|v|}).$$

Thus, in the case $Q = Q_{\mathcal{L}}$,

$$Q(h, U_\gamma) = \text{tr}(A[h]D^2U_\gamma) + hU_\gamma = A^*[h]U_\gamma'' + \frac{a[h] - A^*[h]}{|v|}U_\gamma' + hU_\gamma.$$

In particular, since $U_\gamma' = -\gamma r^{-1}U_\gamma$, $U_\gamma'' = \gamma(\gamma+1)|v|^{-2}U_\gamma$, it follows that

$$Q_{\mathcal{L}}(h, U_\gamma) = U_\gamma \left(\gamma(\gamma+1)A^*[h]|v|^{-2} - \gamma(a[h] - A^*[h])|v|^{-2} + h\right).$$

The thesis follows by noticing that $A^*[h] \leq \frac{1}{3}a[h]$.

For the case $Q = Q_{\mathcal{KS}}$, an analogous computation shows that

$$\begin{aligned} Q(h, U_\gamma) &= U_\gamma \left(-\gamma(1-\gamma)a[h]|v|^{-2} + h\right) \\ &\leq U_\gamma \left(-\tfrac{1}{3}\gamma(1-\gamma)a[h]|v|^{-2} + h\right), \end{aligned}$$

where in the last inequality we used $\gamma \in (0,1)$ and $a[h] \geq 0$. □

4. Short time existence.

In this section, the operator Q denotes either $Q_{\mathcal{L}}$ or $Q_{\mathcal{KS}}$. For some nontrivial interval of existence $[0,T)$, a smooth solution to

$$\begin{cases} \partial_t f &= Q(f,f) \text{ in } \mathbb{R}^3 \times [0,T), \\ f(\cdot,0) &= f_{\text{in}}, \end{cases}$$

will be obtained by taking the limit of a sequence of functions $\{f_k\}_{k\geq 0}$ constructed recursively (as explained further below). The interval of existence $[0,T)$ is maximal in the sense that either $T = \infty$ or else the L^∞ norm of $f(\cdot,t)$ blows up as t approaches T, so the classical solution cannot be extended to a longer time interval.

Remark 4.1. As mentioned in the Introduction, existence and uniqueness of bounded weak solutions to (1.4) has been obtained respectively by Arsenev and Peskov [2] and by Fournier in [6]. It is likely (but not at all obvious) that the method used in [6] will carry over to the case of the isotropic equation (1.7). Thus, for the sake of completeness, we provide in this section a detailed proof of existence (but not uniqueness) of a classical solution for the nonlinear problem that covers the isotropic equation. For completely classical solutions this is certainly new for the isotropic equation (1.7) with $\alpha = 1$, although the methods used in the proof – a priori estimates for linear equations, which yield compactness for a sequence of approximate solutions to the nonlinear problem– are fairly well known, but still somewhat different of the approach used in [11] for the case $\alpha < 3/4$. Uniqueness for classical solutions of (1.4) is contained in Fournier's result [6], since classical solutions are in particular weak solutions, and as it was just mentioned above, it is likely that this result can be expanded to cover the equation (1.7).

For technical reasons we first assume that f_{in} satisfies (1.8) and for some $c > 0$,

$$f_{\text{in}} \in C^{2+\beta}(\mathbb{R}^3), \quad \|f_{\text{in}}\|_{C^{2+\beta}(B_1(v))} \leq \frac{c}{1+|v|^5}, \ \ \forall\, v \in \mathbb{R}^3. \tag{4.1}$$

The last inequality yields a rate of decay for the second derivatives of f_{in} which somewhat simplify the existence proof. The assumptions (4.1) are auxiliary, and will be removed (by an approximation argument) in the proof of the Theorem 4.14 at the end of this section.

Fix $\delta > 0$. A sequence $\{f_k^\delta\}_{k\geq 0}$ will be constructed recursively, so that for every k

$$f_k^\delta \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)) \cap C^{2+\beta,1+\beta/2}(\mathbb{R}^3 \times \mathbb{R}_+), \tag{4.2}$$

for some $\alpha \in (0,1)$ independent of k. The construction is done as follows: first, we set $f_0(v,t) := f_{\text{in}}(v)$ for all v and $t > 0$. Next, assuming we have constructed $f_{k-1}^\delta \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)) \cap C^{2+\beta,1+\beta/2}(\mathbb{R}^3 \times \mathbb{R}_+)$, define f_k^δ as the unique classical solution to the linear Cauchy problem

$$\begin{cases} \partial_t f & = \delta \Delta f + Q(f_{k-1}^\delta, f) \text{ in } \mathbb{R}^3 \times \mathbb{R}_+, \\ f(\cdot,0) & = f_{\text{in}}. \end{cases} \tag{4.3}$$

The fact that the sequence f_k^δ is well defined and satisfies (4.2) follows by repeatedly applying Lemma 2.1, making use of the fact that for every $k \geq 1$, $\beta' \in (0,1)$,

$$f_k^\delta \text{ satisfies (4.2) and solves (4.3)} \Rightarrow A[f_k^\delta], \nabla a[f_k^\delta] \in C^{\beta',\beta'/2}(\mathbb{R}^3 \times [0,\infty)). \tag{4.4}$$

That this is so is essentially a consequence of the fact that $A[f_k^\delta]$ and $\nabla a[f_k^\delta]$ are convolutions of f_k^δ with relatively nice kernels; we do not write out the explicit proof of the above fact here, as the proof is essentially the same as that of Lemma 4.7, where a quantified version of the assertion (4.4) is proved. Thus, we have entirely constructed the sequence $\{f_k^\delta\}_{k\geq 0}$, each f_k^δ being also radially symmetric and monotone, thanks to Proposition 3.1 and (1.8).

Remark 4.2. Note that, for the purpose of iteration in k, the coefficients $A[f_{\text{in}}]$ and $\nabla a[f_{\text{in}}]$ (which are independent of time) are Hölder continuous in space thanks to (4.1).

Once we have constructed the sequence $\{f_k^\delta\}_k$, we will focus in showing that it converges locally uniformly in $\mathbb{R}^3 \times [0, T_*^\delta)$ (δ fixed, $k \to \infty$) to some function f^δ in $\mathbb{R}^3 \times [0,T_*^\delta)$, where f^δ is a classical solution of

$$\partial_t f^\delta = \delta \Delta f^\delta + Q(f^\delta, f^\delta), \ \ f^\delta = f_{\text{in}}$$

The proof of this fact will take most of this section, it is achieved in Theorem 4.12. The selection of T_*^δ will guarantee that either $T_*^\delta = \infty$ or else $\|f^\delta(\cdot,t)\|_\infty$ blows up as $t \to T_*^\delta$. Then, we take the limit $\delta \to 0$ along a subsequence, making sure f^δ and its derivatives converge locally uniformly to a solution of the original nonlinear problem, this is done in Theorem 4.14, where the auxiliary assumption (4.1) is also removed.

We start by using a differential inequality argument to control the L^∞-norm of the f_k^δ uniformly in k and δ for at least some time interval depending only on $\|f_{\text{in}}\|_{L^\infty(\mathbb{R}^3)}$.

Lemma 4.3. *Let $\{f_k^\delta\}_k$ be the sequence defined above. Then, for every $k \in \mathbb{N}$ we have*

$$f_k^\delta(0,t) \leq \frac{f_{\text{in}}(0)}{1-f_{\text{in}}(0)t}, \quad \forall\, t \in \left[0, \frac{1}{f_{\text{in}}(0)}\right).$$

Proof. Since $f_{\text{in}}(0) > 0$ it is immediate that the estimate holds for $k = 0$. Arguing by induction, suppose that

$$f_{k-1}^\delta(0,t) \leq \frac{f_{\text{in}}(0)}{1-f_{\text{in}}(0)t}, \quad \forall\, t \in \left[0, \frac{1}{f_{\text{in}}(0)}\right).$$

Let us prove the corresponding inequality for f_k^δ. By virtue of f_k^δ being smooth, radially symmetric and monotone decaying, it follows that $f_k^\delta(0,t) \geq f_k^\delta(v,t)$ for all v and t and $D^2 f_k^\delta(0,t) \leq 0$ for all t. Plugging this information on the equation solved classical by f_k^δ, we obtain

$$\begin{aligned}\partial_t f_k^\delta(0,t) &= 2^{-k}\Delta f_k^\delta(0,t) + \operatorname{tr}(A[f_{k-1}^\delta](0,t) D^2 f_k^\delta(0,t)) + f_{k-1}^\delta(0,t) f_k^\delta(0,t)\\ &\leq f_{k-1}^\delta(0,t) f_k^\delta(0,t).\end{aligned}$$

Then, we may integrate the following differential inequality in time

$$\partial_t f_k^\delta(0,t) \leq f_{k-1}^\delta(0,t) f_k^\delta(0,t),$$

and it follows that

$$f_k^\delta(0,t) \leq f_{\text{in}}(0) e^{\int_0^t f_{k-1}^\delta(0,s)\,ds} \leq f_{\text{in}}(0) e^{\int_0^t \frac{f_{\text{in}}(0)}{1-f_{\text{in}}(0)s}\,ds}, \quad \forall\, t \in \left[0, \frac{1}{f_{\text{in}}(0)}\right),$$

where the last inequality was due to the inductive hypothesis. Since

$$\int_0^t \frac{f_{\text{in}}(0)}{1-f_{\text{in}}(0)s}\,ds = -\log(1-f_{\text{in}}(0)t),$$

it follows, as desired, that

$$f_k^\delta(0,t) \leq \frac{f_{\text{in}}(0)}{1-f_{\text{in}}(0)t}, \quad \forall\, t \in \left[0, \frac{1}{f_{\text{in}}(0)}\right).$$

$\square$

Continuing with our analysis of the sequence $\{f_k^\delta\}_k$, we introduce a quantity that will play a crucial role in what follows: for every $T > 0, \delta > 0$ let

$$M(f_{\text{in}}, T, \delta) := \sup_k \left\| f_k^\delta \right\|_{L^\infty(\mathbb{R}^3 \times [0,T])} = \sup_k \sup_{0 \leq t \leq T} f_k^\delta(0,t). \tag{4.5}$$

Lemma 4.3 shows that $M(f_{\text{in}}, T, \delta) < \infty$ for at least every $T < f_{\text{in}}(0)^{-1}$ and any $\delta > 0$. For the rest of this section, we will be concerned only with those T's such that

$$M(f_{\text{in}}, T, \delta) < \infty. \tag{4.6}$$

Remark 4.4. In the following series of Lemmas and Propositions, culminating with Theorem 4.12, we will use a series of estimates that will depend on f_{in}, T,δ and the function $M(f_{\text{in}}, T, \delta)$. For the sake of brevity, throughout this section we will write the letters $C(f_{\text{in}}, T, \delta)$, $C_0(f_{\text{in}}, T, \delta)$, $C_1(f_{\text{in}}, T, \delta)$,$C'(f_{\text{in}}, T, \delta)$ (as well as $c(f_{\text{in}}, T, \delta)$ et cetera) to denote constants that depend solely on f_{in},T,δ and $M(f_{\text{in}}, T, \delta)$, with the understanding that the constants may change from one line to the next.

The next proposition says that we can control the L^∞-norm of the coefficients of the equation (4.3) uniformly in k and δ, as long as (4.6) holds.

Proposition 4.5. *Let δ, k be arbitrary and $M(f_{\text{in}}, T, \delta)$ as in (4.5). For any $t \leq T$ and $v \in \mathbb{R}^3$ we have the pointwise bounds*

$$A[f_k^\delta](v,t) \leq a[f_k^\delta](v,t)\mathbb{I} \leq \frac{2(M(f_{\text{in}}, T, \delta)+1))}{1+|v|}\mathbb{I}, \tag{4.7}$$

$$|\nabla a[f_k^\delta](v,t)| \leq \frac{M(f_{\text{in}}, T, \delta)+1}{1+|v|^2}. \tag{4.8}$$

Proof. The bound (4.7) follows immediately from (3.2) in Lemma 3.3 applied to $h = f_k^\delta$. On the other hand, from Newton's formula (3.3) one sees immediately that

$$\nabla a[f_k^\delta] = -\frac{v}{4\pi|v|^3}\int_{B_{|v|}} f_k^\delta(w,t)\, dw. \tag{4.9}$$

Therefore,

$$|\nabla a[f_k^\delta](v,t)| = \frac{1}{4\pi|v|^2}\int_{B_{|v|}} f_k^\delta(w,t)\, dw.$$

Using the fact that $\|f_k^\delta(\cdot,t)\|_{L^1} = 1$ yields

$$|\nabla a[f_k^\delta](v,t)| \leq \frac{1}{4\pi|v|^2} \;\; \forall\, (v,t),$$

while

$$\begin{aligned} |\nabla a[f_k^\delta](v,t)| &\leq \frac{1}{4\pi|v|^2}\frac{4\pi}{3}|v|^3\|f_k^\delta(\cdot,t)\|_{L^\infty} \\ &\leq \frac{1}{3}M(f_{\text{in}}, T, \delta), \;\; \forall\, (v,t) \in B_1(0)\times[0,T]. \end{aligned}$$

Using that $4\pi|v|^2 \geq 1+|v|^2$ if $|v| \geq 1$, we combine the previous inequalities to obtain the bound

$$|\nabla a[f_k^\delta](v,t)| \leq \frac{M(f_{\text{in}}, T, \delta)+1}{1+|v|^2}, \;\; (v,t) \in \mathbb{R}^3\times[0,T],$$

which proves (4.8). □

For the purposes of controlling the size of $f_k^\delta(v,t)$ for large v, it is necessary to bound the second moment of f_k^δ, in a manner which is uniform in k.

Proposition 4.6. *Let $T > 0$ and $\delta \in (0, 1/10)$. For any $k \in \mathbb{N}$, f_k^δ satisfies the bound*

$$\int_{\mathbb{R}^3} f_k^\delta(v,t)|v|^2\, dv \leq 3 + 10\,(1+M(f_{\text{in}}, T, \delta))\,T, \quad \forall\, t \in [0,T]. \tag{4.10}$$

Proof. Let $\phi(v)$ be a smooth function with compact support. Using the equation solved by f_k^δ, and integrating by parts, we obtain for every $t > 0$

$$\frac{d}{dt}\int f_k^\delta(v,t)\phi(v)\, dv = \int f_k^\delta\left(\delta\Delta\phi + \text{tr}(B[f_{k-1}^\delta]D^2\phi) + 2(\nabla a[f_{k-1}^\delta], \nabla\phi)\right)\, dv.$$

Above $B[f_{k-1}^\delta]$ denotes $a[f_{k-1}^\delta]\mathbb{I}$ or $A[f_{k-1}^\delta]$ depending on whether $Q = Q_{\mathcal{KS}}$ or $Q = Q_{\mathcal{L}}$. Integrating in time, it follows that

$$\begin{aligned}
&\int f_k^\delta(v,t_2)\phi(v)\;dv - \int f_k^\delta(v,t_1)\phi(v)\;dv \\
&= \int_{t_1}^{t_2}\int f_k^\delta \left(\delta\Delta\phi + \mathrm{tr}(B[f_{k-1}^\delta]D^2\phi) + 2(\nabla a[f_{k-1}^\delta],\nabla\phi)\right)\;dvdt,
\end{aligned}$$

for all $0 \leq t_1 < t_2$. Next, we apply this identity to the sequence $\phi_j(v) = |v|^2\eta_j(v)$, where $\eta_j \in C_c^\infty(\mathbb{R}^3)$, and $\eta_j(v) \to 1$ locally uniformly. Due to the integrability of f_k^δ and the bounds (4.7)-(4.8), we have enough decay at infinity to pass to the limit $j \to \infty$ in the integral and conclude that the identity also holds for the function $\phi(v) = |v|^2$. Therefore, given $0 \leq t_1 < t_2$ we have the identity,

$$\int f_k^\delta(v,t_2)|v|^2\;dv - \int f_k^\delta(v,t_1)|v|^2\;dv = \int_{t_1}^{t_2}\int f_k^\delta\left(\delta 6 + 2\mathrm{tr}(B[f_{k-1}^\delta]) + 4(\nabla a[f_{k-1}^\delta],v)\right)\;dvdt.$$

Now, the bounds (3.2)-(3.3) guarantee that in $\mathbb{R}^3\times[0,T]$ we have

$$\begin{aligned}
\mathrm{tr}(B[f_{k-1}^\delta]) &\leq 2M(f_{\mathrm{in}},T,\delta)+2,\\
|(\nabla a[f_{k-1}^\delta],v)| &\leq \frac{(M(f_{\mathrm{in}},T,\delta)+1)|v|}{1+|v|^2} \leq M(f_{\mathrm{in}},T,\delta)+1.
\end{aligned}$$

Therefore, as long as $t\in[0,T]$

$$\begin{aligned}
&\left|\int_{t_1}^{t_2}\int f_k^\delta\left(\delta 6 + 2\mathrm{tr}(B[f_{k-1}^\delta]) + 4(\nabla a[f_{k-1}^\delta],v)\right)\;dvdt\right|\\
&\leq \int_{t_1}^{t_2}\int f_k^\delta\left(\delta 6 + 8M(f_{\mathrm{in}},T,\delta)+8\right)\;dvdt\\
&\leq \left(6\delta + 8 + 8M(f_{\mathrm{in}},T,\delta)\right)(t_2-t_1).
\end{aligned}$$

Taking $t_1 = 0$ it follows that for $\delta\in(0,1/10)$

$$\int f_k^\delta(v,t_2)|v|^2\;dv \leq \int f_{\mathrm{in}}|v|^2\;dv + 10\left(1+M(f_{\mathrm{in}},T,\delta)\right)T\;\;\forall\, t\in[0,T].$$

Since $\int f_{\mathrm{in}}|v|^2\;dv = 3$ by assumption (1.8), this proves the proposition. □

Next, we show how $f_{k-1}^\delta \in L^\infty(\mathbb{R}_+,L^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3))\cap C^{\alpha,\alpha/2}(\mathbb{R}^3\times\mathbb{R}_+)$ implies Hölder continuity of the coefficients appearing in $Q(f_{k-1}^\delta,f)$, and emphasizing the estimate is uniform in k for $\delta>0$ fixed whenever T is such that (4.6) holds.

Lemma 4.7. *Let $\delta\in(0,1/10)$ and $T>0$ such that (4.6) holds. Then, there is an absolute constant $C>0$ such that for any $\alpha\in(0,1)$ we have the following bound for every $k\geq 1$,*

$$\begin{aligned}
[A[f_k^\delta]]_{C^{\alpha,\alpha/2}(\mathbb{R}^3\times[0,T])} &\leq C\left([f_k^\delta]_{C^{\alpha,\alpha/2}(\mathbb{R}^3\times[0,T])} + M(f_{\mathrm{in}},T,\delta)+1\right),\\
[\nabla a[f_k^\delta]]_{C^{\alpha,\alpha/2}(\mathbb{R}^3\times[0,T])} &\leq C\left([f_k^\delta]_{C^{\alpha,\alpha/2}(\mathbb{R}^3\times[0,T])} + M(f_{\mathrm{in}},T,\delta)+1\right).
\end{aligned}$$

Proof. Let $\eta\in C^\infty(\mathbb{R}^3)$ be an even function such that $\eta\equiv 1$ in $B_1(0)$ and $\eta\equiv 0$ outside B_2. Let us write,

$$A[f_k^\delta] = A_1[f_k^\delta] + A_2[f_k^\delta].$$

Each A_i $(i = 1, 2)$ is given by convolutions $A_i[f_k^\delta] = K_i * f_k^\delta$ with the respective kernels

$$K_1(v) := \frac{1}{8\pi|v|}\left(\mathrm{Id} - \frac{v \otimes v}{|v|^2}\right)\eta(v), \quad K_2(v) := \frac{1}{8\pi|v|}\left(\mathrm{Id} - \frac{v \otimes v}{|v|^2}\right)(1 - \eta(v)).$$

Let us show that A_1, A_2 are Hölder continuous in v ant t. We shall make use of the fact that there is a constant $C(\eta)$ such that

$$\int_{\mathbb{R}^3} |K_1(v)|\, dv + \sup_v |K_2(v)| + \sum_{i=1}^{3} \sup_v |\partial_i K_2(v)| + \sum_{i,j=1}^{3} \sup_v |\partial_{ij} K_2(v)| \leq C(\eta),$$

where the matrix norm used is the standard L^2 norm $|A| = \mathrm{tr}(AA^*)^{1/2}$. For A_1 it is straightforward that

$$\begin{aligned} |A_1(v_1, t_1) - A_1(v_2, t_2)| &\leq \int_{B_2} |K_1(w)||f_k^\delta(v_1 - w, t_1) - f_k^\delta(v_2 - w, t_2)|\, dw \\ &\leq \left(\int_{B_2} |K_1(w)|\, dw\right) \sup_{w \in B_2(0)} |f_k^\delta(v_1 - w, t_1) - f_k^\delta(v_2 - w, t_2)|, \end{aligned}$$

the above holding for any (v_i, t_i), then

$$[A_1]_{C^{\alpha,\alpha/2}} \leq C(\eta)[f_k^\delta]_{C^{\alpha,\alpha/2}}.$$

Next we deal with A_2, which in fact will be Lipschitz continuous. Fix $e \in \mathbb{S}^2$ and write $K_{2,e}(v) := (K_2(v)e, e)$. Using the equation for f_k^δ and integration by parts,

$$\begin{aligned} \partial_t(A_2[f_k^\delta](v)e, e) = \int_{B_1^c} K_{2,e}(w - v)\partial_t f_k^\delta\, dw &= -\int_{B_1^c} (\nabla_w K_{2,e}(w - v), (A[f_{k-1}^\delta] + \delta\mathbb{I})\nabla_w f_k^\delta)\, dw \\ &\quad + \int_{B_1^c} f_k^\delta (\nabla_w a[f_{k-1}^\delta], \nabla_w K_{2,e}(w - v))\, dw. \end{aligned}$$

Integrating by parts once again,

$$\begin{aligned} -\int_{B_1^c} (\nabla_w K_{2,e}(w - v), (A[f_{k-1}^\delta] + \delta\mathbb{I})\nabla_w f_k^\delta)\, dw &= \int_{B_1^c} \mathrm{div}_w((A[f_{k-1}^\delta] + \delta\mathbb{I}) \cdot \nabla_w K_{2,e}(w - v)) f_k^\delta\, dw \\ &= \int_{B_1^c} f_k^\delta\, \mathrm{tr}(A[f_{k-1}^\delta] D_w^2 K_{2,e}(w - v))\, dw \\ &\quad + \int_{B_1^c} f_k^\delta \nabla_w a[f_{k-1}^\delta] \cdot \nabla_w K_{2,e}(w - v))\, dw \\ &\quad + \delta \int_{B_1^c} f_k^\delta \Delta_w K_{2,e}(w - v)\, dw. \end{aligned}$$

Gathering all of the above, it follows that

$$\begin{aligned} \partial_t(A_2[f_k^\delta]e, e) = &\int_{B_1^c} f_k^\delta\, \mathrm{tr}(A[f_{k-1}^\delta] D_w^2 K_{2,e}(w - v))\, dw + 2\int_{B_1^c} f_k^\delta (\nabla_w a[f_{k-1}^\delta], \nabla_w K_{2,e}(w - v))\, dw \\ &+ \delta \int_{B_1^c} f_k^\delta \Delta_w K_{2,e}(w - v)\, dw. \end{aligned}$$

Therefore, we have the bound

$$\begin{aligned}|\partial_t(A_2[f_k^\delta](v)e,e)| &\leq \|D^2K_{2,e}\|_{L^\infty}\|A[f_{k-1}^\delta]\|_{L^\infty}\|f\|_{L^1} + 2\|\nabla K_{2,e}\|_{L^\infty}\|\nabla a[f_{k-1}^\delta]\|_{L^\infty}\|f\|_{L^1}\\ &\quad + \delta\|\Delta K_{2,e}\|_{L^\infty}\|f_k^\delta\|_{L^1}\\ &\leq \|K_{2,e}\|_{C^2}\left(\|A[f_{k-1}^\delta]\|_{L^\infty} + \|\nabla a[f_{k-1}^\delta]\|_{L^\infty} + \delta\right)\\ &\leq \|K_{2,e}\|_{C^2}(3M(f_{\text{in}},T,\delta)+4),\end{aligned}$$

where we used (4.7)-(4.8) and $\delta \in (0,1/10)$ in the last inequality. Since $\|K_{2,e}\| \leq C(\eta)$ for all e,

$$|\partial_t(A_2[f_k^\delta](v)e,e)| \leq 4C(\eta)(M(f_{\text{in}},T,\delta)+1).$$

This immediately implies a Lipschitz bound in time for A_2, namely

$$|A_2(v,t_1) - A_2(v,t_2)| \leq 12\|K_2\|_{C^2}(M(f_{\text{in}},T,\delta)+1)|t_1-t_2|, \quad \forall\, v\ \in \mathbb{R}^3, \quad t_1,\ t_2 \geq 0.$$

For the spatial regularity, from the definition of A_2 and the triangle inequality it follows that

$$\begin{aligned}|A_2(v_1,t) - A_2(v_2,t)| &\leq \int |K_2(w-v_1) - K_2(w-v_2)| f_k^\delta(w,t)\, dw\\ &\leq C(\eta)|v_1-v_2| \int f_k^\delta(w,t)\, dw \ \ \forall\, v_i \in \mathbb{R}^3,\ t \geq 0.\end{aligned}$$

Then, thanks to $\|f_k^\delta(\cdot,t)\|_{L^1} = 1$, it follows that

$$|A_2(v_1,t) - A_2(v_2,t)| \leq C(\eta)|v_1-v_2|, \ \ \forall\, v_i \in \mathbb{R}^3, t > 0.$$

Finally, we combine the estimates in time and space to see that

$$\begin{aligned}&|A_2(v_1,t_1) - A_2(v_2,t_2)|\\ &\leq |A_2(v_1,t_1) - A_2(v_2,t_1)| + |A_2(v_2,t_1) - A_2(v_2,t_2)|\\ &\leq 15C(\eta)(M(f_{\text{in}},T,\delta)+1)(|v_1-v_2| + |t_1-t_2|), \quad \forall\, (v_i,t_i).\end{aligned}$$

Since $|v_1-v_2| + |t_1-t_2| \leq |v_1-v_2|^\alpha + |t_1-t_2|^{\alpha/2}$ when $|v_1-v_2|, |t_1-t_2| \leq 1$, we conclude that

$$[A_2]_{C^{\alpha,\alpha/2}(\mathbb{R}^3\times[0,T])} \leq 15C(\eta)(M(f_{\text{in}},T,\delta)+1).$$

The proof of Hölder regularity for $\nabla a[f_k^\delta](v,t)$ can be done in an entirely analogous manner, writing the kernel as the sum of respectively integrable and C^2 parts. One may also make a slightly different argument, using the fact that since f_k^δ is spherically symmetric, we have the identity (4.9) which yields a similar bound. □

For the purposes of the proof of existence of solutions, we require several parabolic estimates that are local in space but uniform up to the $t = 0$. Notice these are different to the interior estimates stated in Section 2, namely Theorems 2.2, 2.3 and 2.4, which will be of chief importance in latter sections. The parabolic estimates hold in a space-time cylinder, which starts at time $t = 0$, and are in terms of respective norm of the initial data. They guarantee in particular that under the auxiliary assumptions (4.1) on f_{in} the functions f_k^δ will have spatial decay on their second derivatives.

Lemma 4.8. *(Hölder estimate for regular initial data) There exists some $\alpha \in (0,1)$ and constant c which only depends on δ, f_{in}, T and $[f_{\text{in}}]_{C^{2+\beta}(\mathbb{R}^3)}$ such that for any $v \in \mathbb{R}^3$ and $k \geq 1$*

$$[f_k^\delta]_{C^{\alpha,\alpha/2}(B_1(v)\times[0,T])} \leq c(\delta, M(f_{\text{in}},T,\delta), [f_{\text{in}}]_{C^1(B_2(v))}). \tag{4.11}$$

(Schauder estimate up to the initial time) Let $\beta \in (0,1)$. Then for any $v \in \mathbb{R}^3, k \geq 1$,

$$[f_k^\delta]_{C^{2+\alpha,1+\alpha/2}(B_1(v)\times[0,T])} \leq C(\|f_k^\delta\|_{L^\infty(B_2(v)\times[0,T])} + [f_{\text{in}}]_{C^{2+\beta}(B_2(v))}), \tag{4.12}$$

where $C = C(f_{\text{in}}, T, \delta)$.

Proof. For the proof of the first estimate we refer to [12, Thm. 10.1 Page 204]. Note that the constant does not depend in any way on the regularity of the coefficients in the equation solved by f_k^δ, and depends only on the ellipticity constants, and the regularity of f_{in}. The second estimate follows from [12, Thm 10.1 page 351], noting that the space-time Hölder norm of the coefficients $A[f_{k-1}^\delta], \nabla a[f_{k-1}^\delta]$is bounded by a constant $C(f_{\text{in}}, T, \delta)$, thanks to 4.7 and the first estimate (4.11) applied to f_{k-1}^δ (when $k > 1$, for $k = 1$ $f_0^\delta \equiv f_{\text{in}}$ which is regular in space and constant in time). $\square$

Next we show that the diffusion matrices $A[f_k^\delta] + \delta\mathbb{I}$ are Hölder continuous in a manner which is uniform in k (but possibly depending on δ). In this case, standard estimates for linear parabolic equations will yield Hölder bounds on the second order spatial derivates and first order temporal derivatives for f_k^δ, these being uniform in k. Particularly, since we are assuming a spatial decay for the second derivatives of f_{in} (see (2.2)) the same will hold for f_k^δ.

Proposition 4.9. *Let $\delta \in (0, 1/10)$ and $0 < T < \infty$ be such that (4.6) holds. Then there is a constant $C = C(\delta, f_{\text{in}}, T)$ such that for any $v \in \mathbb{R}^3$, Moreover, there is a C depending only on f_{in}, δ, T, $M(f_{\text{in}}, T, \delta)$ such that*

$$\|D^2 f_k^\delta\|_{C^\alpha(B_1(v)\times[0,T])} \leq C(1 + |v|^5)^{-1} \;\; \forall\, v \in \mathbb{R}^3. \tag{4.13}$$

Proof. We will first show that $f_k^\delta(v,t)$ decays as $(1 + |v|^5)^{-1}$ for v large. For that fix $v \in \mathbb{R}^3$, then the spherically symmetry and radial monotonicity of f_k^δ implies that

$$\begin{aligned}\frac{7}{6}\pi|v|^3 f_k^\delta(v,t) &\leq \int_{B_{|v|}\setminus B_{|v|/2}} f_k^\delta(w,t)\, dw \\ &\leq \frac{4}{|v|^2}\int_{\mathbb{R}^3} f_k^\delta(w,t)|w|^2\, dw.\end{aligned}$$

Using the second moment bound (4.10) we arrive at the estimate

$$f_k^\delta(v,t) \leq \frac{4}{\pi|v|^5}\left(3 + 10(1 + M(f_{\text{in}}, T, \delta))T\right)$$

for all $|v| \geq 1$ and $t \in [0,T]$. Since $f_k^\delta(v,t) \leq M(f_{\text{in}}, T, \delta)$ as long as $t \leq T$ we conclude that

$$f_k^\delta(v,t) \leq \frac{C'(f_{\text{in}}, T, \delta)}{1 + |v|^5}, \;\; \forall\, (v,t) \in \mathbb{R}^3 \times [0,T], \tag{4.14}$$

with $C'(f_{\text{in}}, T, \delta) := \max\{M, \frac{4}{\pi}(3 + 10(1 + M(f_{\text{in}}, T, \delta))T)\}$. The bound follows combining the initial bound (4.1), the decay estimate(4.14) and the estimate (4.12) from Lemma 4.8. $\square$

So far we have shown the existence of the sequence $\{f_k^\delta\}$, and proven several uniform estimates which are uniform k for times $T < T_*^\delta$. Moving towards obtaining a limit from this sequence, we prove an iterative estimate on the size of the functions $\{f_k^\delta - f_{k-1}^\delta\}_k$ in $\mathbb{R}^3 \times [0,T]$, for $\delta > 0$ fixed and T such that $C(f_{\text{in}}, T, \delta) < \infty$.

Lemma 4.10. *Let $\delta \in (0, 1/10)$ and $T > 0$ be such that (4.6) holds and let $w_k^\delta := f_{k-1}^\delta - f_k^\delta$ for each $k \geq 1$. There is a number $0 < T_0 < T$, $T_0 = T_0(f_{\text{in}}, T, \delta)$ such that*

(1) *For each $k \geq 2$,*

$$\|w_k(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[0,T_0])} \leq \frac{1}{4}\|w_{k-1}(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[0,T_0])}.$$

(2) *For each $k \geq 2$, and $l = 1, ..., l_0$ we have*

$$\begin{aligned}\|w_k^\delta(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[t_{l-1},t_l])} \leq \frac{1}{4}&\|w_{k-1}^\delta(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[t_{l-1},t_l])}\\ &+ 2\|w_k^\delta(t_{l-1})\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}.\end{aligned}$$

Here $l_0 \in \mathbb{N}$ is the largest such that $(l_0-1)T_0 \leq T$ and $t_l := \min\{lT_0, T\}$.

Proof. We shall drop the superscript δ for convenience. Using the respective equations for f_{k-1} and f_k we get that $w_k = f_{k-1} - f_k$ satisfies

$$\begin{aligned}\partial_t w_k &= \delta\Delta w_k + \mathrm{tr}(A[f_{k-2}]D^2 w_k) + f_{k-2}w_k\\ &\qquad + \mathrm{tr}(A[w_{k-1}]D^2 f_k) + f_k w_{k-1}, \quad \text{for } t > 0,\\ w_k &= 0 \quad \text{for } t = 0.\end{aligned} \tag{4.15}$$

Step 1. According to Proposition 4.9, there is a positive constant $C(f_{\text{in}}, T, \delta)$ such that

$$|D^2 f_k^\delta(v,t)| \leq C(f_{\text{in}}, T, \delta)(1+|v|^5)^{-1}, \;\; \forall\, v \in \mathbb{R}^3, t \in [0,T]. \tag{4.16}$$

The estimate (4.16) and the estimate (3.5) applied to w_{k-1} imply the inequality

$$|\mathrm{tr}(A[w_{k-1}]D^2 f_k(v,t))| \leq C(f_{\text{in}}, T, \delta)\left(\frac{\|w_{k-1}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} + \|w_{k-1}(\cdot,t)\|_{L^1(\mathbb{R}^3)}}{1+|v|^5}\right),$$

which holds for any $(v,t) \in \mathbb{R}^3 \times [0,T]$. On the other hand, $\langle v\rangle^{-4} \in L^1(\mathbb{R}^3)$, therefore

$$\|w_k(t)\|_{L^1} = \int_{\mathbb{R}^3} |w_k(v,t)|\langle v\rangle^4\langle v\rangle^{-4}\, dv \leq \|w_k(t)\langle v\rangle^4\|_{L^\infty}\|\langle v\rangle^{-4}\|_{L^1(\mathbb{R}^3)}.$$

Substituting this in the last estimate, we arrive at the bound,

$$|\mathrm{tr}(A[w_{k-1}]D^2 f_k(v,t))| \leq C(f_{\text{in}}, T, \delta)\|w_{k-1}(t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}(1+|v|^5)^{-1}.$$

Step 2. Consider the function $h_0(v) := \langle v\rangle^{-4} = (1+|v|^2)^{-2}$, then

$$\begin{aligned}Dh_0(v) &= -4(1+|v|^2)^{-3}\, v,\\ D^2 h_0(v) &= -4(1+|v|^2)^{-3}\mathbb{I} + 24(1+|v|^2)^{-4} v\otimes v.\end{aligned}$$

In particular,

$$\begin{aligned}\Delta h_0 &= 12\left(|v|^2-1\right)\langle v\rangle^{-8},\\ \mathrm{tr}(A[f_{k-2}]D^2 h_0) &= -4\langle v\rangle^{-6} a[f_{k-2}] + 24\langle v\rangle^{-8}(A[f_{k-2}]v, v).\end{aligned}$$

Using the inequalities $||v|^2-1|, |v|^2 \leq \langle v\rangle^2$, the above leads to

$$\begin{aligned}|\delta\Delta h_0| &\leq 12\delta\langle v\rangle^{-6},\\ |\mathrm{tr}(A[f_{k-2}]D^2 h_0| &\leq 4\langle v\rangle^{-6} a[f_{k-2}] + 24\langle v\rangle^{-6} a[f_{k-2}].\end{aligned}$$

Then, recalling that $\delta \in (0, 1/10) \Rightarrow 12\delta < 3/2$ we combine the above inequalities into one,

$$|\delta\Delta h_0 + \mathrm{tr}(A[f_{k-2}]D^2 h_0)| \leq 28\,(1 + a[f_{k-2}])\,\langle v\rangle^{-6} \leq 56(1 + C(f_{\text{in}}, T, \delta))h_0,$$

where we have used (4.7) to bound $a[f_{k-2}]$.

Step 3. Next, let

$$H_0(v,t) := RA^{-1}(e^{At}-1)h_0(v),$$

for $A, R > 0$ to be determined. It is immediate that

$$\partial_t H_0 = AH_0 + Rh_0.$$

The last inequality in *Step 2* implies that

$$|\delta\Delta H_0 + \mathrm{tr}(A[f_{k-2}]D^2H_0)| + f_{k-2}H_0 \leq 60(1+C(f_{\mathrm{in}},T,\delta))H_0.$$

The estimates from *Step 1*, the definition of $h_0(v)$ and (4.14) yield

$$\mathrm{tr}(A[w_{k-1}]D^2 f_k) + f_k w_{k-1} \leq C_0\|w_{k-1}(t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}h_0(v),$$

with $C_0 = C_0(f_{\mathrm{in}},T,\delta)$. In light of this, for any $T_0 \in (0,T)$, we choose A and R as follows

$$\begin{aligned} A &= 60(1+C(f_{\mathrm{in}},T,\delta)),\\ R &= C_0 \sup_{0\leq t\leq T_0}\|w_{k-1}(t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}. \end{aligned}$$

In which case, we have for any $(v,t)\in\mathbb{R}^3\times[0,T_0]$

$$\begin{aligned} \partial_t H_0 &\geq 60(1+C(f_{\mathrm{in}},T,\delta))H_0 + C_0\left(\|w_{k-1}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} + \|w_{k-1}(\cdot,t)\|_{L^1(\mathbb{R}^3)}\right)h_0\\ &\geq \delta\Delta H_0 + \mathrm{tr}(A[f_{k-2}]D^2H_0) + f_{k-2}H_0 + \left(\mathrm{tr}(A[w_{k-1}]D^2f_k) + f_k w_{k-1}\right). \end{aligned}$$

This means that H_0 is a supersolution of (4.15), the parabolic equation solved by w_k. Moreover $H_0(\cdot,0) = w_k(\cdot,0) = 0$. Then, thanks to the comparison principle

$$w_k \leq H_0 \text{ in } \mathbb{R}^3\times[0,T_0].$$

The same argument applied to $-w_k$ yields,

$$\eta_k \leq H_0 \text{ in } \mathbb{R}^3\times[0,T_0].$$

We have shown there are constants $C_0(f_{\mathrm{in}},T,\delta)$ and $C_1(f_{\mathrm{in}},T,\delta)$ such that

$$|w_k(v,t)| \leq C_0\|w_{k-1}(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[0,T_0])}(e^{C_1(f_{\mathrm{in}},T,\delta)t}-1)\langle v\rangle^{-4}, \text{ in } \mathbb{R}^3\times[0,T_0].$$

In particular, there is a T_0, depending only on T and $C_0(f_{\mathrm{in}},T,\delta)$, such that

$$T_0\in(0,T), \text{ and } C_0(e^{C_1(f_{\mathrm{in}},T,\delta)T_0}-1) \leq \frac{1}{4}.$$

This results on the estimate,

$$\|w_k(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[0,T_0])} \leq \frac{1}{4}\|w_{k-1}(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[0,T_0])},$$

and the first part of the lemma is proved.

Step 4. Fix $k\geq 2$. Assume for now that $2T_0 < T$ –same T_0 as in Step 3– and define the function $H_1 : \mathbb{R}^3\times[T_0,\infty)\to\mathbb{R}$ by

$$H_1(v,t) := RA^{-1}(e^{A(t-T_0)}-1)h_0(v) + \|w_k(T_0)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}h_0(v),$$

where A and R are to be determined. A straightforward computation yields

$$\begin{aligned}\partial_t H_1 = Re^{A(t-t_0)}h_0(v) &= A\Big(RA^{-1}(e^{A(t-T_0)}-1)h_0(v) + \|w_k(T_0)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}h_0(v).\Big)\\
&\quad + Rh_0(v) - \|w_k(T_0)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}h_0(v)\\
&= AH_1 + Rh_0(v) - \|w_k(T_0)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}h_0(v).\end{aligned}$$

As in the previous step, we have

$$\begin{aligned}&\delta\Delta H_1 + \mathrm{tr}(A[f_{k-2}]D^2H_1) + f_{k-2}H_1 + \mathrm{tr}(A[w_{k-1}]D^2 f_k) + f_k w_{k-1} + 2^{-k}\Delta f_k\\
&\leq 60(1+C(f_{\mathrm{in}},T,\delta))H_1 + h_0 C_0\|w_{k-1}(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[T_0,2T_0])}\\
&= AH_1 + h_0(R - A\|w_k(v,T_0)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}) = \partial_t H_1,\end{aligned}$$

by choosing

$$\begin{aligned}A =& 60(1+C(f_{\mathrm{in}},T,\delta)),\\
R =& C_0\|w_{k-1}(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[T_0,2T_0])} + 60(1+C(f_{\mathrm{in}},T,\delta))\|w_k(v,T_0)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}.\end{aligned}$$

Likewise, $H_1(\cdot,T_0) \geq w_k(\cdot,T_0)$. Then, just as before, the comparison principle says that $H_1(\cdot,t) \geq w_k(\cdot,t)$ for $t \in [T_0,2T_0]$,

$$\begin{aligned}|w_k(v,t)| \leq\ & C_0(e^{C_1(f_{\mathrm{in}},T,\delta))t}-1)\|w_{k-1}(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[T_0,2T_0])}\langle v\rangle^{-4}\\
&+ \|w_k(v,T_0)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}(e^{C_1(f_{\mathrm{in}},T,\delta)t}-1)\langle v\rangle^{-4}\\
&+ \|w_k(v,T_0)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3)}\langle v\rangle^{-4}.\end{aligned}$$

Hence for $t \in [T_0,2T_0]$ we get

$$\begin{aligned}\|w_k(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[T_0,2T_0])} \leq\ & \frac{1}{4}\|w_{k-1}(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[T_0,2T_0])}\\
&+ 2\|w_k(v,T_0)(1+|v|^2)^2\|_{L^\infty(\mathbb{R}^3)}.\end{aligned}$$

This yields the second estimate, in the case $l=2$. The above argument can be repeated to obtain a respective estimate in the interval $[2T_0,3T_0]$, and so on. After a finite number of iterations we will reach some $l_0 \in \mathbb{N}$ such that $(l_0-1)T_0 \leq T$ and $l_0T_0 > T$. In that case we repeat the above argument on the interval $[(l_0-1)T_0,T]$, yielding the respective bound and completing the proof of the second estimate.

□

The next lemma shows that if $\delta \in (0,1/10)$ and T is a time for which (4.6) holds, the sequence f_k^δ will converge uniformly in $\mathbb{R}^3\times[0,T]$ to a continuous limit f^δ.

Lemma 4.11. *Let $\{f_k^\delta\}_k$, $\delta \in (0,1/10)$, and $T>0$ be such that (4.6) holds. Then there is a continuous function $f^\delta : \mathbb{R}^3\times[0,T]\to\mathbb{R}$ such that*

$$\begin{aligned}\lim_k \|f^\delta - f_k^\delta\|_{L^\infty(\mathbb{R}^3\times[0,T])} = 0,\\
\lim_k \|f^\delta - f_k^\delta\|_{L^\infty(0,T;L^1(\mathbb{R}^3))} = 0.\end{aligned}$$

Proof. Let $T_0>0$, l_0 and t_l be as in Lemma 4.10. Define, for $l = 0,1,\ldots,l_0$ and $k\in\mathbb{N}$

$$E_{k,l} := \|w_k(v,t)\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3\times[t_{l-1},t_l])}.$$

Then, Lemma 4.10 says the following recursive relations hold for $k \geq 2$ and $l = 0, \dots, l_0$

$$E_{k,1} \leq \frac{1}{4} E_{k-1,1},$$
$$E_{k,l} \leq 4E_{k,l-1} + \frac{1}{2} E_{k-1,l}.$$

We claim that these recurrence relations guarantee the summability in k of the sequence $\{E_{k,l}\}_k$ for any fixed $l = 1, \dots, l_0$. The first recurrence relation implies that $E_{k,1}$ decays geometrically, thus we immediately have

$$\sum_{k=3}^{\infty} E_{k,1} < \infty.$$

Next, suppose that for some $1 < l < l_0$ we have

$$\sum_{k=3}^{\infty} E_{k,l} < \infty.$$

Taking the sum for k from 3 to N of the second recursive relation we get

$$\sum_{k=3}^{N} \frac{1}{2} E_{k,l+1} \leq 4 \sum_{k=3}^{N} E_{k,l} + \frac{1}{2} E_{2,l+1}.$$

We can then pass to the limit $N \to +\infty$, and use the summability for $E_{k,l}$ to obtain

$$\sum_{k=3}^{N} \frac{1}{2} E_{k,l+1} < +\infty.$$

Combining the summability of the sequences $\{E_{k,l}\}_k$ for every $l \leq l_0$, we conclude that

$$\sum_k \|(f_k(v,t) - f_{k-1}(v,t))\langle v\rangle^4\|_{L^\infty(\mathbb{R}^3 \times [0,T])} < \infty.$$

Since $\langle v\rangle \geq 1$ for all v, and $\langle v\rangle^{-4} = (1+|v|^2)^{-2} \in L^1(\mathbb{R}^3)$, this implies that

$$\sum_k \|f_k - f_{k-1}\|_{L^\infty(\mathbb{R}^3 \times [0,T]))} < \infty,$$
$$\sum_k \|f_k - f_{k-1}\|_{L^\infty(0,T;L^1(\mathbb{R}^3))} < \infty.$$

This summability implies $\{f_k\}$ is a Cauchy sequence in each norm, proving the lemma. □

Theorem 4.12. *For each $\delta \in (0, 1/10)$ there is a time $T_*^\delta = T_*^\delta(f_{\text{in}})$ with $0 < T_*^\delta \leq \infty$ and a function f^δ in $C^{2,1}_{\text{loc}}\left(\mathbb{R}^3 \times [0, T_*)\right)$, such that*

$$\begin{cases} \partial_t f^\delta & = \delta \Delta f^\delta + Q(f^\delta, f^\delta) \text{ in } \mathbb{R}^3 \times [0, T_*^\delta), \\ f^\delta(\cdot, 0) & = f_{\text{in}}. \end{cases}$$

Moreover, either $T_*^\delta = \infty$ or

$$\limsup_{T \to T_*^{\delta -}} \|f^\delta\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} = \infty.$$

Proof. Step 1. Let

$$T_*^\delta := \sup\{T>0 \;\mid\; M(f_{\rm in},T,\delta)<\infty\}.$$

By Lemma 4.3 we have $T_*^\delta \geq (2f_{\rm in}(0))^{-1}$, thus $T_*^\delta > 0$. It may certainly be that $T_*^\delta = \infty$. Now, we may apply Lemma 4.11 to f_k^δ and any fixed $T < T_*^\delta$, resulting in a continuous function $f^\delta : \mathbb{R}^3 \times [0,T_*^\delta) \to \mathbb{R}$ such that

$$f_k^\delta \to f^\delta \text{ uniformly in } \mathbb{R}^3\times[0,T),\;\; \forall\, T<T_*^\delta.$$

On the other hand, we have the estimates from Lemma 4.8 which guarantee, by the Arzela-Ascoli theorem, that for any subsequence $k_n \to \infty$ there is a subsequence k_n' such that $\partial_t f_{k_n'}^\delta$ and $D^2 f_{k_n'}^\delta$ converge locally uniformly in $\mathbb{R}^3\times[0,T_*)$ as $n\to\infty$. Since $f_k^\delta \to f$ locally uniformly and $\{k_n\}$ was arbitrary it follows that (i) $f^\delta \in C^{2,1}_{\rm loc}\left(\mathbb{R}^3\times[0,T_*)\right)$, and (ii) the sequences $D^2 f_k^\delta$ and $\partial_t f_k^\delta$ converge locally uniformly to $D^2 f^\delta$ and $\partial_t f^\delta$ as $k\to\infty$, respectively.

Step 2. Let us show the matrices $\{A[f_k^\delta]\}_k$ converge locally uniformly in $\mathbb{R}^3\times[0,T_*^\delta)$ to $A[f^\delta]$. Indeed, let $t\in[0,T_*^\delta)$ and apply the estimate (3.5) to $g = |f_k(\cdot,t)-f_k^\delta(\cdot,t)|$ (which is a non-negative, bounded, spherically symmetric function), which leads to the bound

$$|A[f_k^\delta](v,t)-A[f^\delta](v,t)| \leq 2\left(\|f_k(\cdot,t)-f_k^\delta(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}+\|f_k(\cdot,t)-f_k^\delta(\cdot,t)\|_{L^1(\mathbb{R}^3)}\right)$$

for all v and $t<T_*^\delta$. Then Lemma 4.11 shows $A[f_k^\delta]$ converges uniformly to $A[f^\delta]$ uniformly in $\mathbb{R}^3\times[0,T]$ for every $T<T_*^\delta$.

Step 3. Thanks to the local uniform convergence of $f_k^\delta, D^2 f_k^\delta, \partial_t f_k^\delta$ and $A[f_k^\delta]$ proved in the previous two steps, we can pass to the limit in the equation for f_k^δ and conclude that

$$\partial_t f^\delta = \delta\Delta f^\delta + Q(f^\delta,f^\delta) \text{ in } \mathbb{R}^3\times[0,T_*^\delta).$$

Step 4. We show here that if T_*^δ is finite, then the L^∞ norm of $f^\delta(\cdot,t)$ goes to infinity as t approaches T_*^δ. Arguing by contradiction, suppose that T_*^δ is finite, and

$$\limsup_{T\to T_*^\delta} f^\delta(0,t) < +\infty.$$

Since f^δ is continuous and bounded for any $t<T_*^\delta$ then f^δ is bounded for any $t\leq T_*^\delta$ and in particular

$$f^\delta(0,T_*^\delta-\varepsilon)\leq C, \quad \varepsilon>0.$$

The uniform convergence of $f_k^\delta\to f^\delta$ for all $t<T_*^\delta$ shows that for any small enough $\varepsilon>0$ there is some k_0 such that

$$f_k^\delta(0,T_*^\delta-\varepsilon) < 2C,\;\; \forall\, k>k_0. \tag{4.17}$$

Since $\sup_k f_k^\delta(0,T_*^\delta-\varepsilon)<+\infty$, then (4.17) implies

$$f_k^\delta(0,T_*^\delta-\varepsilon) < \tilde{C},\;\; \forall\, k\geq 1.$$

Then, the differential inequality argument from Lemma 4.3, applied with starting time shifted to $T_*^\delta-\varepsilon$, proves that

$$f_k^\delta(0,T_*^\delta-\varepsilon+t) \leq \frac{\tilde{C}}{1-\tilde{C}t},\;\; \forall\, k\geq 1,\; 0<t<\frac{1}{\tilde{C}}.$$

Taking now $t = \frac{1}{2\tilde{C}}$ and $\varepsilon = \frac{1}{4\tilde{C}}$ yields

$$f_k^\delta(0, T_*^\delta + \varepsilon) < 2\tilde{C},$$

which contradicts the maximality of T_*^δ. The theorem is proved.

□

Next, we show that as long as $f^\delta(v,t)$ is bounded in a time internal $[0,T]$, the mass of $f^\delta(v,t)$ cannot escape to infinity nor concentrate at the origin. The bound is independent of δ. A consequence of this result is a local lower bound for $A[f^\delta]$ along radial directions.

Proposition 4.13. *Let $\delta \in (0, 1/10)$, f^δ be a function given by Theorem 4.14, $T < T_*^\delta$ and let $M > 0$ be such that*

$$\|f^\delta\|_{L^\infty \times [0,T]} < M.$$

Then, there are radii $r(f_{\text{in}}, T, M)$ and $R(f_{\text{in}}, T, M)$ such that $0 < r < R < \infty$ and

$$\int_{B_R \setminus B_r} f^\delta(v,t)\, dv \geq \frac{1}{2}, \quad \forall\, t \in [0,T]. \tag{4.18}$$

As a consequence, there is a positive constant $c_0 = c_0(f_{\text{in}}, T, M)$ such that

$$A^*[f^\delta](v,t) \geq \frac{c_0}{1+|v|^3}, \quad \forall\, v \in \mathbb{R}^3, t \in [0,T],\ k \in \mathbb{N}, \tag{4.19}$$

where $A^*[\cdot]$ is as defined in (3.2).

Proof. Given $R > 0$, the mass of f^δ outside $B_R(0)$ may be estimated via its second moment

$$\int_{B_R^c} f^\delta\, dv \leq \int_{B_R^c} f^\delta \frac{|v|^2}{R^2}\, dv \leq \frac{1}{R^2} \int_{\mathbb{R}^3} f^\delta(v,t)|v|^2\, dv.$$

Moreover for any r, R with $R > r > 0$ there is the obvious lower bound,

$$\begin{aligned}
\int_{B_R \setminus B_r} f^\delta(v,t)\, dv &= 1 - \int_{B_R^c} f^\delta(v,t)\, dv - \int_{B_r} f^\delta(v,t)\, dv \\
&\geq 1 - \frac{1}{R^2} \int_{\mathbb{R}^3} f^\delta(v,t)|v|^2\, dv - \frac{4\pi}{3} r^3 M.
\end{aligned} \tag{4.20}$$

where we made use of the fact that $\|f^\delta(\cdot,t)\|_{L^1} = 1$. Following exactly the same steps as in the proof of Proposition 4.6 one can show

$$\int_{\mathbb{R}^3} f^\delta(v,t)|v|^2\, dv \leq 3 + 10\,(1+M)\,T, \quad \forall\, t \in [0,T]. \tag{4.21}$$

Hence (4.18) follows from (4.20) and (4.21) by choosing

$$\begin{aligned}
R &:= 2\left(3 + 10(1+M)T\right)^{1/2}, \\
r &:= (8\pi M)^{-1/3}.
\end{aligned}$$

Finally (4.19) follows from (4.18), the selection of R and r above, and Lemma 3.3.

□

Theorem 4.14. *Given $f_{\rm in}$ as in (1.8), there is a time T_* and a function $f \in C^{2;1}_{\rm loc}\left(\mathbb{R}^3\times(0,T_*)\right)$ with initial data $f_{\rm in}$, which solves (1.4) or (1.7). Moreover, either $T_*=\infty$ or*

$$\limsup_{t\to T_*^-}\|f\|_{L^\infty(\mathbb{R}^3\times[0,t])}=\infty.$$

The initial data is achieved in the sense that for any $\phi\in C^\infty_c(\mathbb{R}^3)$ and any $t\in(0,T_*)$ we have

$$\int_{\mathbb{R}^3} f(v,t)\phi(v)\;dv-\int_{\mathbb{R}^3} f_{\rm in}(v)\phi(v)\;dv=-\int_0^t\int(B[f]\nabla f-f\nabla a[f],\nabla\phi)\;dv\;dt.$$

Here $B[f]$ denotes $A[f]$ or $a[f]\mathbb{I}$ depending on whether we are dealing with (1.4) or (1.7).

Proof. Step 1. Let us assume first that $f_{\rm in}$ satisfies the additional assumptions (4.1), this assumption will be removed in the final step. For each $n\in\mathbb{N}$, let $f_n:=f^{\delta_n}$ and $T_n:=T_*^{\delta_n}$ correspond to f^δ with $\delta=10^{-n}$, as constructed in Theorem 4.12. Then, each f_n is a spherically symmetric, monotone solution to

$$\partial_t f_n=\tfrac{1}{10^n}\Delta f_n+Q(f_n,f_n)\;\text{ in }\mathbb{R}^3\times[0,T_n),\;\; f_n(v,0)=f_{\rm in}(v).$$

Moreover, for each n, we have that either $T_n=\infty$ or else $\|f_n(\cdot,t)\|_\infty\to\infty$ as $t\to T_n$.

We define T_* by

$$T_*:=\inf\{T\mid \liminf_n M(f_{\rm in},T,10^{-n})=\infty\},\tag{4.22}$$

with the understanding that $T_*=\infty$ if the set above is empty. As before, it is not difficult to see that $T_*\geq(2f_{\rm in}(0))^{-1}$. See Remark 4.15 for further discussion about the definition of T_*.

Step 2. Let us show then that there exists a solution in $\mathbb{R}^3\times(0,T_*)$. Let T_j be a strictly increasing sequence of times, with $\lim T_j=T_*$. Fix j, then since $T_j<T_*$ there is a subsequence $\{n_{j,k}\}$, $n_{j,k}\to\infty$ as $k\to\infty$, and such that

$$\sup_k M(f_{\rm in},T,10^{-n_{j,k}})<\infty.$$

The above combined with Proposition 4.13 implies there is a constant $c=c(f_{\rm in},T_j)$ such that for all $k\in\mathbb{N}$ we have

$$A[f_{n_{j,k}}](v,t)\geq\frac{c(f_{\rm in},T_j)}{1+|v|^3}\mathbb{I},\;\;\forall\,(v,t)\in\mathbb{R}^3\times(0,T_j).$$

The interior Hölder estimate (Theorem 2.2), then says that for any cylinder $Q\subset\subset\mathbb{R}^3\times(0,T)$ we have

$$[f_{n_{j,k}}]_{C^{\alpha,\alpha/2}(Q)}\leq C(Q,T_j),\;\;\forall\,k.$$

From here, the same argument as in Lemma 4.7 shows that $A[f_{n_{j,k}}]$ and $\nabla a[f_{n_{j,k}}]$ are $C^{\alpha,\alpha/2}$ uniformly in k in compact subsets of $\mathbb{R}^3\times(0,T_j)$. Accordingly, the uniform regularity of these coefficients together with the Schauder estimates (Theorem 2.4) guarantee that for every cylinder $Q\subset\subset\mathbb{R}^3\times(0,T_j)$ we have a constant $C(Q,T_j)$ independent of k such that

$$[f_{n_{j,k}}]_{C^{2+\alpha,1+\alpha/2}(Q)}\leq C(Q,T_j).$$

Then, the Arzela-Ascoli theorem and a Cantor diagonalization argument yield local uniform convergence of f_n to a function f in $\mathbb{R}^3\times(0,T)$ which will be differentiable in time, and second order differentiable in space. In particular, $\tilde f_j$ is a spherically symmetric, monotone solution to

$$\partial_t\tilde f_j=Q(\tilde f_j,\tilde f_j)\text{ in }\mathbb{R}^3\times(0,T_j),\;\;\tilde f_j(\cdot,0)=f_{\rm in},$$

with f_{in} as in (1.8). We can take this argument one step further, and apply the Arzela-Ascoli one more time to the sequence $\{\tilde{f}_j\}_j$ and conclude that along a subsequence they (along with their derivatives) converge uniformly in compact subsets of $\mathbb{R}^3 \times (0, T_*)$ to a function

$$f : \mathbb{R}^3 \times (0, T_*) \to \mathbb{R}$$

which is again a solution. In summary: we have constructed a function $f : \mathbb{R}^3 \times (0, T_*)$ which is differentiable in time and second order differentiable in space, such that

$$\partial_t f = Q(f, f)$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^3} f(v,t)\phi(v)\, dv - \int_{\mathbb{R}^3} f_{\text{in}}(v)\phi(v)\, dv \\
&= -\int_0^t \int (B[f]\nabla f - f\nabla a[f], \nabla\phi)\, dv\, dt \;\; \forall \phi \in C_c^\infty(\mathbb{R}^3),\; t \in (0, T_*). \qquad (4.23)
\end{aligned}$$

Moreover, the function f has the property that for every $T < T_*$, there is a sequence $n_k \to \infty$ such that the functions f_{n_k} defined in Step 1 converge to f locally uniformly in $\mathbb{R}^3 \times [0, T]$.

Step 3. It remains to show that if $T_* < \infty$, then the solution built in Step 2 blows up in L^∞ as time approaches T_*. We argue by contradiction, similarly to the proof of Theorem 4.12, but with a slight modifications accounting for the fact that we do not know whether the functions f_n have a unique limit as $n \to \infty$ (see Remark 4.15 for further discussion). Suppose $C > 0$ is a constant such that

$$\lim_{T \to T_*} \|f\|_{L^\infty(\mathbb{R}^3 \times [0,T])} < C.$$

Let $\varepsilon > 0$ be a small number (to be determined), according to Step 2, there is a sequence $n_k \to \infty$ such that $f_{n_k} \to f$ locally uniformly in $\mathbb{R}^3 \times [0, T_* - \varepsilon/2]$. In particular, there must be some $k_0 > 0$ such that

$$\|f_{n_k}\|_{L^\infty(B_1 \times [0, T_* - \varepsilon])} < 2C, \;\; \forall\, k > k_0.$$

As in the proof of Theorem 4.14, choosing ε such that $2\varepsilon(2C) < 1/2$, the differential inequality argument guarantees that

$$\|f_{n_k}\|_{L^\infty(\mathbb{R}^3 \times [0, T_* + \varepsilon])} \leq 4C, \;\; \forall\, k > k_0.$$

This shows there is a positive $\varepsilon > 0$ such that

$$\liminf_n M(f_{\text{in}}, T, 10^{-n}) < \infty \;\; \forall\, T < T_* + \varepsilon.$$

This is impossible, since T_* is the infimum of $\{T \mid \liminf_n M(f_{\text{in}}, T, 10^{-n}) = \infty\}$. This contradiction shows that

$$\lim_{T \to T_*} \|f\|_{L^\infty(\mathbb{R}^3 \times [0,T])} = \infty,$$

and the theorem is proved at least for f_{in} for which (4.1) holds.

Step 5. In order to remove (4.1), given f_{in} for which only (1.8) holds let $f_{\text{in}}^{(n)}$ be a sequence of functions such that (4.1) holds for each $f_{\text{in}}^{(n)}$ (with a constant c that may depend on n) and such that

$$\lim_n \|f_{\text{in}} - f_{\text{in}}^n\|_{L^\infty} = \lim_n \|f_{\text{in}} - f_{\text{in}}^n\|_{L^1} = 0.$$

Let $f^{(n)}$ be a corresponding sequence of solutions as constructed in Steps 1-4 above, then each $f^{(n)}$ is defined up to some time $T_{*,n}$. The times $T_{*,n}$ are bounded uniformly away from 0 since $f_{\text{in}} \in L^\infty$. The functions $f^{(n)}$ enjoy uniform local a priori estimates, therefore the same compactness argument from Step 2 allow us to pick a subsequence $n_k \to \infty$ and a time T_* such that the functions $f^{(n_k)}$ and their derivatives have a local uniform limit as $k \to \infty$ to a function $f : \mathbb{R}^3 \times (0, T_*) \to \mathbb{R}$ which is smooth solution to the nonlinear equation and which blows up in L^∞ as time approaches T_*. Finally, fixing a test function ϕ and $t \in (0, T_*)$ we may apply (4.23) to each $f^{(n_k)}$ and conclude that f satisfies the respective relation in the limit, proving the theorem.

□

Remark 4.15. It is worth comparing the definition of T_*^δ in Theorem 4.12 with that of T_* in Theorem 4.14. In the present situation, a priori it is unclear whether the sequence f_n has a unique limit as $n \to \infty$. Hence, if we define

$$T_* := \sup\{T \mid \sup_n M(f_{\text{in}}, T, 10^{-n}) < \infty\},$$

the existence of a subsequence bounded for times strictly greater then T^* does not contradict the definition of T^*. The contradiction however holds if T^* is defined via the lim inf as in (4.22). In the proof of the former theorem, matters were simplified by the fact that $\{f_k^\delta\}_k$ was a Cauchy sequence (for δ fixed), meaning in particular that if it is shown that a subsequence of f_k^δ remains bounded in $[0, T]$, then the entire sequence remains bounded. This was key in proving the maximality of the interval of existence $(0, T_*^\delta)$.

5. Pointwise bounds and proof of Theorem 1.1

5.1. **Conditional pointwise bound.** The first lemma of this section (Lemma 5.2) is the key argument for the proofs of Theorem 1.1 and Theorem 1.3. It consists of a barrier argument based on the observation that the function $U(v) = |v|^{-\gamma}$ is a supersolution for the elliptic operator $Q(f, \cdot)$ under certain assumptions on f (this is where the radial symmetry and monotonicity is needed). It affords control of certain spatial L^p-norms of the solution, and from these higher regularity will follow by standard elliptic estimates (Lemma 5.5).

First, we prove an elementary proposition that will be of use in proving the key lemma.

Proposition 5.1. *If h is a non-negative, radially symmetric and decreasing function,*

$$\frac{h(v)}{a[h](v)} \leq 8 \sup_{r \leq |v|} \left\{ r^2 \frac{\int_{B_r} h(w)\, dw}{\int_{B_r} a[h](w)\, dw} \right\} |v|^{-2}, \quad \forall\, v \in \mathbb{R}^3.$$

Proof. First off, since h is radially symmetric and decreasing,

$$\frac{1}{|B_{|v|}(0)|} \int_{B_{|v|}(0)} h(w)\, dw \geq h(v).$$

On the other hand, since $h \geq 0$ and (in particular) $a[h]$ is superharmonic,

$$\begin{aligned} a[h](v) &\geq \frac{1}{|B_{2|v|}(v)|} \int_{B_{2|v|}(v)} a[h](w)\, dw, \\ &= \frac{2^{-3}}{|B_{|v|}(0)|} \int_{B_{|v|}} a[h](w)\, dw, \quad \forall\, v \in \mathbb{R}^3. \end{aligned}$$

Therefore,

$$\frac{h(v)}{a[h](v)} \leq 8\frac{\int_{B_{|v|}} h(w)\;dw}{\int_{B_{|v|}(0)} a[h](w)\;dw},$$

which implies that

$$\frac{h(v)}{a[h](v)} \leq 8|v|^{-2}\sup_{r\leq |v|}\left\{ r^2\frac{\int_{B_r} h(w)\;dw}{\int_{B_r} a[h](w)\;dw}\right\}.$$

□

Lemma 5.2. *Suppose $f:\mathbb{R}^3\times[0,T]\to\mathbb{R}_+$ is a classical solution of (2.1). Let $\gamma\in(0,1)$, suppose there exists some $R_0>0$ such that*

$$r^2\frac{\int_{B_r} g(w,t)\;dw}{\int_{B_r} a[g](w,t)\;dw} \leq \tfrac{1}{24}\gamma(1-\gamma),\;\;\forall\, r\leq R_0,\;\; t\leq T. \tag{5.1}$$

Then,

$$f(v,t)\leq \max\left\{\tfrac{3}{4\pi}R_0^{\gamma-3},(\tfrac{3}{4\pi})^{\gamma/3}\|f_{\mathrm{in}}\|_{L^{3/\gamma}_{\mathrm{weak}}}\right\}|v|^{-\gamma},\;\;\text{in } B_{R_0}\times[0,T].$$

In particular, the conclusion of the lemma holds for some $R_0>0$ whenever there is a modulus of continuity $\omega(r)$ and some $R_1>0$ such that

$$\sup_{r<|v|}\sup_{t\in[0,T]}\left\{r^2\frac{\int_{B_r} g(w,t)\;dw}{\int_{B_r} a[g](w,t)\;dw}\right\}\leq \omega(|v|),\;\;\forall\, 0<|v|\leq R_1. \tag{5.2}$$

Remark 5.3. It is easy to see that for any radially decreasing function $h(v)$ the condition that h belongs to $L^p_{\mathrm{weak}}(\mathbb{R}^3)$ implies that h lies below a power function of the form $1/|v|^{3/p}$, and vice versa. More precisely,

$$\|h(v)\|_{L^p_{\mathrm{weak}}}\leq C \Leftrightarrow h(v)\leq C\left(\tfrac{3}{4\pi}\right)^{1/p}|v|^{-3/p}. \tag{5.3}$$

Proof. Let $U_\gamma=|v|^{-\gamma}$, then Proposition 3.4 says that

$$Q(g,U_\gamma)\leq U_\gamma a[g]\left(-\tfrac{1}{3}\gamma(1-\gamma)|v|^{-2}+g/a[g]\right).$$

Applying Proposition 5.1 with $h=g(\cdot,t)$,

$$\frac{g}{a[g]}(v,t)\leq 8|v|^{-2}\sup_{r\leq|v|}\left\{r^2\frac{\int_{B_r} g(w,t)\;dw}{\int_{B_r} a[g](w,t)\;dw}\right\}\leq \tfrac{1}{3}\gamma(1-\gamma)|v|^{-2},$$

where we used (5.1) to get the last inequality. It follows that

$$Q(g,U_\gamma)\leq 0,\quad \text{in } B_{R_0}\times[0,T]. \tag{5.4}$$

In particular, if there is a modulus of continuity as in (5.2), then $Q(g,U_\gamma)\leq 0$ in $B_{R_0}\times[0,T]$ provided R_0 is chosen so that $\omega(R_0)\leq 1/24$.

On the other hand, given that $f(v,t)$ is radially decreasing and lies in L^1 (see Remark 5.3),

$$f(v,t)\leq \tfrac{3}{4\pi|v|^3}\|f\|_{L^1(\mathbb{R}^3)}=\tfrac{3}{4\pi|v|^3},\;\;\forall\, v\in\mathbb{R}^3,\; t\in[0,T]. \tag{5.5}$$

Where we used that $\|f(\cdot,t)\|_{L^1(\mathbb{R}^3)}=1$ for all t. Finally, the function $\tilde{U}_\gamma(v)$ defined by

$$\tilde{U}_\gamma(v):=\max\left\{\tfrac{3}{4\pi}R_0^{\gamma-3},(\tfrac{3}{4\pi})^{\gamma/3}\|f_{\mathrm{in}}\|_{L^{3/\gamma}_{\mathrm{weak}}}\right\}|v|^{-\gamma},$$

is a supersolution for the equation solved by f in $B_{r_0}\times[0,T]$. Moreover, clearly $\tilde{U}_\gamma$ lies above f_{in} in B_{R_0}, while by (5.5) $\tilde{U}_\gamma$ lies above f in $\partial B_{R_0}\times[0,T]$. Then, the comparison principle implies that $f\leq \tilde{U}_\gamma$ in $B_{r_0}\times[0,T]$, and the lemma is proved. □

The next lemma deals specifically with solutions to the nonlinear equations, (1.4) or (1.7). It controls from below the integral of a solution in some ball B_R. For the case of the Landau equation (1.4) the constant is independent of time (by conservation of mass and second moment), while for the Krieger-Strain equation (1.7) the bound decays exponentially in time.

Lemma 5.4. *For f solving (1.4), there is a constant $R>0$ such that*

$$\int_{B_R} f(v,t)\;dv \geq 1/2, \qquad t>0. \tag{5.6}$$

For f solving (1.7), and any radii $R>r>0$ there are $\beta>0$ and $C_0>0$ such that

$$\int_{B_R\setminus B_r} f(v,t)\;dv \geq C_0e^{-\beta t}\int_{B_{4R}\setminus B_{r/4}} f_{\text{in}}(v)\;dv \quad t>0. \tag{5.7}$$

Proof. If f solves (1.4), then

$$\int_{B_R(0)^c} f(v,t)\;dv \leq R^{-2}\int_{B_R(0)^c} f(v,t)|v|^2\;dv \leq 3R^{-2}.$$

Thus

$$\int_{B_R(0)} f(v,t)\;dv = 1-\int_{B_R(0)^c} f(v,t)\;dv \geq 1-3R^{-2}.$$

Estimate (5.6) follows by choosing R large enough. The corresponding estimate (5.7) for f solving (1.7) follows a similar argument used in [11] and the derivation of the estimate is done in detail in the Appendix. □

The next lemma says that any solution f to (1.4) or (1.7) is a bounded function for all times provided f satisfies (5.2).

Lemma 5.5. *Let $f:\mathbb{R}^3\times[0,T]\to\mathbb{R}$ be a radially symmetric, radially decreasing solution to (1.4) (or (1.7)) with initial data as in (1.8) and such that for some $R_0>0$ we have*

$$r^2\frac{\int_{B_r} f(w,t)\;dw}{\int_{B_r} a[f](w,t)\;dw} \leq \tfrac{1}{24}\gamma(1-\gamma), \;\; \forall\, r\leq R_0, \;\; t\leq T.$$

Or, assume that there is some modulus of continuity $\omega(r)$ such that,

$$\sup_{r<|v|}\sup_{t\in[0,T]}\left\{ r^2\frac{\int_{B_r} f(w,t)\;dw}{\int_{B_r} a[f](w,t)\;dw}\right\} \leq \omega(|v|), \;\; \forall\, 0<r\leq R_0. \tag{5.8}$$

Then,

$$\sup_{t\in[\frac{1}{2}T,T]} \|f(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \leq C_0. \tag{5.9}$$

For some constant C_0 depending only on f_{in}, T and R_0.

Proof. The assumptions of the lemma are simply the same as those of Lemma 5.2 with $g(v,t) = f(v,t)$, from where it follows (using also Remark 5.3) that

$$\sup_{t\in[0,T]} \|f(\cdot,t)\|_{L^p_{\text{weak}}(B_{R_0})} \leq \max\left\{ \tfrac{3}{4\pi} R_0^{-3(1-1/p)}, (\tfrac{3}{4\pi})^{1/p} \|f_{\text{in}}\|_{L^p_{\text{weak}}} \right\} \||v|^{-3/p}\|_{L^p_{\text{weak}}}$$
$$=: C_0(f_{\text{in}}, R_0, p),$$

for some $p > 6$. By interpolation and the Sobolev embedding it follows that $\|f(\cdot,t)\|_{L^6(\mathbb{R}^3)}$ and $\|\nabla a[f(\cdot,t)]\|_{L^\infty(\mathbb{R}^3)}$ are bounded by some constant C determined by $C_0(f_{\text{in}}, R_0, p)$. Then, applying (2.5) from Theorem 2.3 with $Q = B_{R_0} \times [0,T]$ we arrive respectively at

$$\|f\|_{L^\infty(B_{R_0/2}\times[T/2,T])} \leq C\left\{ \|f\|_{L^2(Q)} + R_0^2 \|\nabla a[f]\|_{L^\infty(Q)} \right\} < \infty,$$

for some $C = C(f_{\text{in}}, R_0, T)$, and the lemma is proved. $\square$

Proof of Theorem 1.2. According to Theorem 4.12, for $f_{\text{in}} \in L^\infty$, there exists a time $T_0 > 0$, and a solution $f(v,t)$ to (1.4) defined in $\mathbb{R}^3 \times [0,T_0)$ and with initial values f_{in}.

The time T_0 is maximal in the sense that $T_0 = \infty$ or else,

$$\lim_{t\to T_0^-} \|f(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} = \infty. \tag{5.10}$$

Moreover, since $f \in L^\infty$ for in $\mathbb{R}^3 \times [0,t]$ for every $t < T_0$, interior regularity estimates (see Theorem 2.2 and Theorem 2.4) show that f must be twice differentiable in v and differentiable in t as long as $t \in (0,T)$.

Finally, arguing by contradiction let us assume that

$$\limsup_{r\to 0^+} \sup_{t\in(0,T_0)} \left\{ r^2 \frac{\int_{B_r} f(v,t)\,dv}{\int_{B_r} a[f](v,t)\,dv} \right\} < 1/96.$$

In this case, there must be some $R_0 > 0$ such that

$$\sup_{t\in(0,T_0)} \left\{ r^2 \frac{\int_{B_r} f(v,t)\,dv}{\int_{B_r} a[f](v,t)\,dv} \right\} \leq 1/96 \;\; \forall\, r \leq R_0.$$

This means Lemma 5.5 can be applied with $T = T_0$, and it follows that

$$\sup_{t\in[\frac{1}{2}T_0,T_0]} \|f(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} < \infty,$$

which is in contradiction with (5.10), and the theorem is proved. $\square$

Proof of Theorem 1.1. As in the proof of Theorem 1.2 we have a solution $f(v,t)$ defined up to some maximal time T_0. In case $T_0 < \infty$, we know that $\|f(\cdot,t)\|_{L^\infty}$ goes to infinity as $t \to T_0^-$. As before, this $f(v,t)$ is twice differentiable in v and differentiable in t for $t \in (0,T)$.

Now, assume the $L^{3/2}$ norm of $f(\cdot,t)$ does not concentrate at 0 as $t \to T^-$. That is, suppose there is a modulus of continuity $\omega(\cdot)$ such that

$$\sup_{t\in(0,T_0)} \|f(\cdot,t)\|_{L^{3/2}(B_r)} \leq \omega(r).$$

Then, there is some $C>0$ such that

$$r^2\frac{\int_{B_r} f(v,t)\;dv}{\int_{B_r} a[f](v,t)\;dv} = \frac{\frac{4\pi}{3r}\int_{B_r} f(v,t)\;dv}{\frac{1}{|B_r|}\int_{B_r} a[f](v,t)\;dv},$$
$$\leq C\frac{1}{r}\int_{B_r} f(v,t)\;dv,\;\;\forall\, r>0, t\in(0,T_0).$$

Then, Hölder's inequality says that

$$r^2\frac{\int_{B_r} f(v,t)\;dv}{\int_{B_r} a[f](v,t)\;dv} \leq C'\|f(\cdot,t)\|_{L^{3/2}(B_r)},$$
$$\leq C'\omega(r).$$

It follows, that if $R_0>0$ is chosen so that $C'\omega(R_0)<1/96$, that Lemma 5.5 can be applied to conclude again that

$$\sup_{t\in[\frac{1}{2}T_0,T_0]}\|f(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}<\infty,$$

which as before directly contradicts $\lim\limits_{t\to T_0^-}\|f(\cdot,t)\|_{L^\infty}=\infty$, and the theorem is proved. □

To end this section, we present a computation indicating that for an arbitrary function f the quotient appearing in the assumption of Theorem 1.2 is always smaller than or equal to 3.

Proposition 5.6. *Let $h\in L^1(\mathbb{R}^3)$ be a nonnegative function. Then,*

$$r^2\frac{\int_{B_r} h(v)\;dv}{\int_{B_r} a[h](v)\;dv}\leq 3,\;\;\forall\, r>0.$$

Remark 5.7. It could be of use in understanding the blow up or (no blow up) of (1.4) to characterize those h for which the above quotient goes to zero as r approaches 0. In particular, it would be useful to understand this when h is not necessarily in a regular enough L^p space or Morrey space, namely when h is such that

$$h\not\in L^{3/2}_{\text{loc}}\quad\text{or}\quad\sup_{r>0}\frac{1}{r}\int_{B_r} h\;dv=\infty.$$

Proof. Let us write $a(v)=a[h](v)$ for the sake of brevity. Note that

$$\int_{B_r} a(v)\;dx=\int_{\mathbb{R}^3} a(v)\chi_B(v)\;dv=\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} h(w)|v-w|^{-1}\chi_{B_r}(v)\;dwdv.$$

The goal is to compare the two integrals,

$$\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} h(w)|v-w|^{-1}\chi_{B_r}(v)\;dwdv,\quad r^2\int_{\mathbb{R}^3} h(v)\chi_{B_r}(v)\;dv.$$

Note that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} h(w)|v-w|^{-1}\chi_{B_r}(v)\;dwdv=\int_{\mathbb{R}^3} h(v)(\chi_{B_r}*\Phi)(v)\;dv,\;\;\Phi(v)=(4\pi|v|)^{-1}.$$

It is not hard to compute $\Phi_B:=\chi_{B_r}*\Phi$ directly. Indeed, it is the unique $C^{1,1}$ solution of

$$\Delta\Phi_{B_r}=-\chi_{B_r},\;\;\Phi_{B_r}\to 0\text{ at }\infty,$$

which has the simple expression,

$$\Phi_{B_r}(x) = \begin{cases} -\frac{1}{6}|v|^2 + \frac{1}{2}r^2 & \text{in } B_r, \\ \frac{1}{3}r^3|v|^{-1} & \text{in } B_r^c. \end{cases}$$

It follows that,

$$\begin{aligned} \int_{B_r} a(v)\,dv &= \int_{B_r} \left(\frac{1}{2}r^2 - \frac{1}{6}|v|^2\right) h(v)\,dv + \frac{r^3}{3}\int_{B_r^c} h(v)|v|^{-1}\,dv, \\ &\geq \int_{B_r} \left(\frac{1}{2}r^2 - \frac{1}{6}|v|^2\right) h(v)\,dv. \end{aligned}$$

This proves the stated bound since the last inequality guarantees that

$$\int_{B_r} a(v)\,dv \geq \frac{r^2}{3}\int_{B_r} h(v)\,dv.$$

□

6. Mass comparison and proof of Theorem 1.3

In this section we apply the ideas from previous sections to construct global solutions (in the radial, monotone case) for equation (1.7), namely

$$\partial_t f = a[f]\Delta f + f^2.$$

In view of Lemma 5.5, the fact that $T_0 = \infty$ in Theorem 1.1 results from a bound of any $L^p(\mathbb{R}^3)$-norm of f, with $p > 3/2$. For (1.7) the bound of any $L^p(\mathbb{R}^3)$-norm of f, with $p > 3/2$ will be proven by a barrier argument done at the level of the *mass function* of $f(v,t)$, which is defined by

$$M_f(r,t) = \int_{B_r} f(v,t)\,dv, \quad (r,t) \in \mathbb{R}_+ \times (0,T_0).$$

Depending on which problem f solves, the associated function $M_f(r,t)$ solves a one-dimensional parabolic equation with diffusivity given by $A^*[f]$ or $a[f]$.

Proposition 6.1. *Let f be a solution of (1.4) (resp. (1.7)) in $\mathbb{R}^3 \times [0,T_0]$, then $M(r,t)$ solves*

$$\partial_t M_f = A^*\partial_{rr}M_f + \frac{2}{r}\left(\frac{M_f}{8\pi r} - A^*\right)\partial_r M_f \quad \text{in } \mathbb{R}_+ \times (0,T_0) \tag{6.1}$$

$$\left(\text{resp. } \partial_t M_f = a\partial_{rr}M_f + \frac{2}{r}\left(\frac{M_f}{8\pi r} - a\right)\partial_r M_f \quad \text{in } \mathbb{R}_+ \times (0,T_0)\right). \tag{6.2}$$

Proof. We briefly show how to obtain (6.2); for (6.1) calculations are identical. Using the divergence theorem and the divergence expression in (1.7) we get

$$\partial_t M_f = \int_{\partial B_r} (a[f]\nabla f - f\nabla a[f], n)\,d\sigma = 4\pi r^2\left(a[f]\partial_r f - f\partial_r a[f]\right).$$

Furthermore, straightforward differentiation yields the formulas

$$4\pi r^2\partial_r f = r^2\partial_r\left(r^{-2}\partial_r M_f\right), \quad \partial_r a[f] = -(4\pi r^2)^{-1}M_f.$$

Substituting these in the expression for $\partial_t M_f$ above we get

$$\partial_t M_f = a[f]r^2\partial_r\left(\frac{1}{r^2}\partial_r M_f\right) + \frac{1}{4\pi r^2}M_f\partial_r M_f.$$

Expansion and rearrangement of the terms result in:

$$\begin{aligned}\partial_t M_f &= a\left(-\frac{2}{r}\partial_r M_f + \partial_{rr} M_f\right) + \frac{M_f}{4\pi r^2}\partial_r M_f\\ &= a\partial_{rr}M_f + \frac{2}{r}\left(\frac{M_f}{8\pi r} - a\right)\partial_r M_f,\end{aligned}$$

and the thesis follows. □

Define the linear parabolic operator L in $\mathbb{R}_+ \times (0,T)$ as

$$Lh := \partial_t h - a\partial_{rr} h - \frac{2}{r}\left(\frac{M_f}{8\pi r} - a[f]\right)\partial_r h.$$

The above proposition simply says that $LM_f = 0$ in $\mathbb{R}_+ \times (0,T)$. The next proposition identifies suitable supersolutions for L.

Proposition 6.2. *If $m \in [0,2]$ and $h(r,t) = r^m$ then $Lh \geq 0$ in $\mathbb{R}_+ \times (0,T)$.*

Proof. By direct computation we see that

$$Lh = -mr^{m-2}\left[(m-1)a + 2\left(\frac{M_f}{8\pi r} - a[f]\right)\right].$$

On the other hand,

$$a[f](r) = \frac{1}{4\pi r}\int_{B_r} f\, dv + \int_{B_r^c} \frac{f}{4\pi |v|}\, dv \geq \frac{M_f}{4\pi r},$$

which guarantees that $\frac{1}{2}a[f](r) \geq \frac{M_f}{8\pi r}$. Thus,

$$\begin{aligned}Lh &= mr^{m-2}\left[(1-m)a[f] + 2\left(a[f] - \frac{M_f}{8\pi r}\right)\right]\\ &\geq mr^{m-2}((2-m)a[f] \geq 0.\end{aligned}$$

The last inequality being true for $m \leq 2$. □

Proof of Theorem 1.3. Assume $f_{\text{in}} \in L^\infty$, in which case Theorem 4.12 yields a solution $f(v,t)$ that exists for some time $T_0 > 0$ (possibly infinite). As the bound for $f(v,t)$ will not rely on the L^∞ norm of f_{in} but a L^p_{weak} norm of f_{in} the existence of a solution for unbounded initial data in L^p ($p > 6$) will follow by a standard density argument.

Since $p > 6$, there is some $\alpha > 0$ and some $C_0 > 0$ (depending only on $\|f\|_{L^p_{\text{weak}}}$) such that

$$M_{f_{\text{in}}}(r,0) = \int_{B_r} f_{\text{in}}\, dv \leq C_0 r^{1+\alpha}.$$

Moreover, since $f(\cdot,t)$ has total mass 1 for every $t > 0$, we also have

$$M_f(r,t) \leq 1, \;\; \forall\, r > 0, t \in (0,T).$$

Proposition 6.2 says that $h = Cr^{1+\alpha}$ is a supersolution of the parabolic equation solved by M_f in $\mathbb{R}_+ \times (0,T)$. Then, choosing $C := \max\{C_0, 1\}$ the comparison principle yields

$$M_f(r,t) \leq h(r) = Cr^{1+\alpha} \quad \text{for } r \in (0,1),\; t \in (0,T). \tag{6.3}$$

Since $f(v,t)$ is radially symmetric and decreasing, bound (6.3) implies that $f(|v|,t) \leq \frac{3C}{4\pi}\frac{1}{|v|^{2-\alpha}}$ for $v \in B_1$ and $t \in (0,T)$; hence there is some $p' > 3/2$ and some $C_{p'} > 0$ such that

$$\|f(\cdot,t)\|_{L^{p'}(B_1)} \leq C_{p'}, \;\; \forall\, t \in (0,T).$$

Then Lemma 5.2 says that $f(v,t)$ is bounded in $\mathbb{R}^3 \times (0,T_0)$. By Lemma 5.5 and the characterization of T_0 in Theorem 4.12, it follows that $T_0 = +\infty$ so the solution is global in time. $\square$

The method of the proof for Theorem 1.3 fails short in preventing finite time blow up for (1.4). In any case, it at least gives another criterium for blow-up, the proof of which is essentially the same as that of Theorem 1.3.

Corollary 6.3. *Suppose that for all $t \in [0,T_0]$ there is some $r_0 > 0$ and $0 < \lambda < 8\pi$ such that*

$$M_f(r,t) \le \lambda r A^*(r,t) \;\; \forall \;\; r < r_0.$$

Then any solution to (1.4) is bounded for any $t > 0$.

7. Appendix

Proof of Proposition 3.1. The radial symmetry of any solution f to (2.1) follows by the uniqueness property of (2.1) and by the fact that $Q(g,f)$ commutes with rotations, as shown below. We first rewrite the collision operator as

$$Q(g,f) = \operatorname{div}(A[g]\nabla f - f\nabla a[g]) = a[g]\Delta f - \operatorname{div}(\tilde{A}[g]\nabla f) + fg,$$

with $\tilde{A}[g]\nabla f := \int \frac{g(|v-y|)}{|y|^3}\langle \nabla f(v), y\rangle y\, dy$.

Let $\mathbb{T}$ be a rotation operator. Since g is radially symmetric, so is $a[g]$. Hence

$$a[g]\Delta(f\circ\mathbb{T}) = a[g\circ\mathbb{T}]\Delta(f\circ\mathbb{T}) = (a[g]\circ\mathbb{T})(\Delta f\circ\mathbb{T}) = (a[g]\Delta f)\circ\mathbb{T},$$

taking into account that the Laplacian operator commutes with rotations. Moreover

$$\begin{aligned}
\operatorname{div}(\tilde{A}[g]\nabla f(\mathbb{T}v)) &= \operatorname{div}\left(\int \frac{g(|v-y|)}{|y|^3}\langle \nabla f(\mathbb{T}v), y\rangle y\, dy\right)\\
&= \operatorname{div}\left(\int \frac{g(|v-y|)}{|y|^3}\langle \mathbb{T}^*\nabla_z f(z)_{|_{z=\mathbb{T}v}}, y\rangle y\, dy\right)\\
&= \operatorname{div}\left(\int \frac{g(|\mathbb{T}(v-y)|)}{|y|^3}\langle \nabla_z f(z)_{|_{z=\mathbb{T}v}}, \mathbb{T}y\rangle \mathbb{T}^*\mathbb{T}y\, dy\right)\\
&= \operatorname{div}\left(\mathbb{T}^*\underbrace{\int \frac{g(|\mathbb{T}v-y)|)}{|y|^3}\langle \nabla_z f(z)_{|_{z=\mathbb{T}v}}, y\rangle y\, dy}_{=:V(\mathbb{T}v)}\right)\\
&= Tr(\mathbb{T}^* Jac(V)_{|_{z=\mathbb{T}v}}\mathbb{T}) + \underbrace{\nabla(Tr(\mathbb{T}^*))}_{=0}\cdot V(\mathbb{T}v)\\
&= Tr(\mathbb{T}\mathbb{T}^* Jac(V)_{|_{z=\mathbb{T}v}})\\
&= Tr(\mathbb{I}d\; Jac(V)_{|_{z=\mathbb{T}v}})\\
&= \operatorname{div}\left(\int \frac{g(|z-y|)}{|y|^3}\langle \nabla_z f(z), y\rangle y\, dy\right)\circ\mathbb{T}.
\end{aligned}$$

Hence $Q(g, f(\mathbb{T}v)) = Q(g,f)\circ\mathbb{T}$.
Now we rewrite the linear equation (2.1) in spherical coordinates:

$$\partial_t f = A^*\partial_{rr} f + \tfrac{a-A^*}{r}\partial_r f + fg, \tag{7.1}$$

with $A^*[g](v) := (A[g](v)\hat{v},\hat{v})$, $\hat{v} := \frac{v}{|v|}$ and differentiate (7.1) with respect to r. The function $w := \partial_r f$ satisfies the following inequality:

$$\partial_t w \leq A^*\partial_{rr}w + \tfrac{a-A^*}{r}\partial_r w + wg + \partial_r A^*\partial_r w + \partial_r\left(\tfrac{a-A^*}{r}\right)w.$$

If $w(\cdot,0)\leq 0$ it follows from maximum principle that $w(\cdot,t)\leq 0$ for all $t\geq 0$. In other words, the (negative) sign of $\partial_r f$ is preserved in time.

□

Proof of Proposition 3.2. The identity (3.3) is a classical and a proof can be found in [13][Section 9.7]. To prove (3.2), let $v\in\mathbb{R}^3$ non-zero, $r := |v|$, then

$$(A[g](v)\hat{v},\hat{v}) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\frac{1}{|v-w|}g(w)\left((\mathbb{I}-\tfrac{v-w}{|v-w|}\otimes\tfrac{v-w}{|v-w|})\hat{v},\hat{v}\right)\,dw.$$

Note that

$$\left((\mathbb{I}-\tfrac{v-w}{|v-w|}\otimes\tfrac{v-w}{|v-w|})\hat{v},\hat{v}\right) = 1-\cos(\hat{\theta}(w))^2,$$

where $\hat{\theta}$ denotes the angle between $w-v$ and v. Consider, for $0\leq t,r$, the function

$$I(r,t) := \int_{\partial B_t}\frac{1-\cos(\hat{\theta})^2}{|v-w|}\,dw.$$

The function $I(r,t)$ encodes all the information about A^*. In particular, integration in spherical coordinates yields the expression

$$A^*[h](v) = \frac{1}{8\pi}\int_0^\infty f(t)I(|v|,t)\,dt.$$

As it turns out, $I(r,t)$ has rather different behavior according to whether $r<t$ or not. By averaging in the v variable, it is not hard to see that

$$I(r,t) = \frac{t^2}{r^4}I(t,r),\ \ \forall\, r<t.$$

Accordingly, we focus on $I(r,t)$ when $r>t$. To do so, denote by θ the angle between w and v and observe that

$$1-\cos(\hat{\theta})^2 = \sin(\hat{\theta})^2 = \frac{t^2-t^2\cos(\theta)^2}{|v-w|^2} = \frac{t^2-w_1^2}{|v-w|^2},$$

where $w_1 = (w,\hat{v})$. Thus,

$$\begin{aligned}
I(r,t) &= \int_{\partial B_t}\frac{t^2-w_1^2}{|v-w|^3}\,dw\\
&= \int_{\partial B_t}\frac{t^2-w_1^2}{(t^2-w_1^2+(r-w_1)^2)^{3/2}}\,dw\\
&= \int_{\partial B_t}\frac{t^2-w_1^2}{(t^2-2rw_1+r^2)^{3/2}}\,dw\\
&= \int_{\partial B_1}\frac{t^2(1-z_1^2)}{t^3(1-2(\frac{r}{t})z_1+(\frac{r}{t})^2)^{3/2}}\,t^2dz\\
&= \int_{\partial B_1}\frac{1-z_1^2}{(1-2(\frac{r}{t})z_1+(\frac{r}{t})^2)^{3/2}}\,tdz.
\end{aligned}$$

This surface integral can be written entirely as an integral in terms of the variable $z_1 \in (-1,1)$,

$$I(r,t) = 2\pi t \int_{-1}^{1} \frac{1-z_1^2}{(1-2(\frac{r}{t})z_1 + (\frac{r}{t})^2)^{3/2}}\,dz_1.$$

For brevity, set for now $s = r/t$, then

$$\begin{aligned}\int_{-1}^{1} \frac{1-z_1^2}{(1-2sz_1+s^2)^{3/2}}dz_1 &= \frac{-2s^4+2s^3+2s-2}{3s^3\sqrt{s^2-2s+1}} - \frac{-2s^4-2s^3-2s-2}{3s^3\sqrt{s^2+2s+1}}\\ &= \frac{-2s^4+2s^3+2s-2}{3s^3(s-1)} - \frac{-2s^4-2s^3-2s-2}{3s^3(s+1)}\\ &= \frac{-2s^4+2s^3+2s-2}{3s^3(s-1)} + \frac{2s^4+2s^3+2s+2}{3s^3(s+1)}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{-2s^4+2s^3+2s-2}{3s^3(s-1)} + \frac{2s^4+2s^3+2s+2}{3s^3(s+1)} &= \frac{2}{3s^3}\left(\frac{-s^4+s^3+s-1}{s-1} + \frac{s^4+s^3+s+1}{s+1}\right)\\ &= \frac{2}{3s^3}\frac{(-s^4+s^3+s-1)(s+1)+(s^4+s^3+s+1)(s-1)}{s^2-1}\\ &= \frac{2}{3s^3}\frac{2s^2-2}{s^2-1} = \frac{4}{3s^3}.\end{aligned}$$

Then, since $s = r/t$, we conclude that

$$\begin{aligned}I(r,t) &= 8\pi\frac{t^4}{3r^3}, \quad \text{for } t<r,\\ I(r,t) &= 8\pi\frac{1}{3t}, \quad \text{for } t>r.\end{aligned}$$

Going back to $A^*[h]$, the above leads to

$$\begin{aligned}A^*[h](v) &= \int_0^r h(t)I(r,t)\,dt + \int_r^\infty h(t)I(r,t)\,dt\\ &= \frac{1}{3r^3}\int_0^r h(t)t^4\,dt + \frac{1}{3}\int_r^\infty h(t)t\,dt.\end{aligned}$$

□

Proof of Lemma 5.4. This argument is inspired by the one in Section 2.6 in [11]. For β, R, r (with $0<r<R$, $0<\beta$) consider the function

$$\Phi(v,t) := e^{-\beta t}(|v|-R)^2(|v|-r)^2.$$

Since Φ is a $C^{1,1}$ function with compact support, it holds

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} f(v,t)\Phi(v)\,dv &= -\int_{\mathbb{R}^3}(a\nabla f - f\nabla a, \nabla\Phi)\,dv\\ &= \int_{\mathbb{R}^3} f\,\mathrm{div}(a\nabla\Phi)\,dv + \int_{\mathbb{R}^3} f(\nabla a,\nabla\Phi)\,dv.\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{div}(a\nabla\Phi)+(\nabla a,\nabla\Phi) &= a\Delta\Phi+2(\nabla a,\nabla\Phi)\\ &= a\Phi''+\frac{2}{|v|}\left(a+|v|a'\right)\Phi'\\ &= a\Phi''+\frac{2}{|v|}\Phi'\int_{|v|}^{+\infty} sf(s,t)\,ds.\end{aligned}$$

It holds:

$$\begin{aligned}&\Phi'(s)=2(R-s)(s-r)(-(s-r)+R-s)=2(R-s)(s-r)(R+r-2s),\\ &\Phi''(s)=2(R-s)(r+R-2s)-2(s-r)(r+R-2s)-4(R-s)(s-r),\\ &\Phi'(r)=\Phi'(R)=0,\ \ \Phi''(r)=\Phi''(R)=2(R-r)^2,\\ &|\Phi''|,|\Phi'|\leq C_{\delta,r,R}\Phi,\quad |v|\in((1+\delta)r,(1-\delta)R).\end{aligned}$$

Hence in a small neighborhood of $|v|=R$ and $|v|=r$ one can show that $\frac{d}{dt}\int_{\mathbb{R}^3} f(v,t)\Phi(v)\,dv\geq 0$; more precisely it holds

$$\operatorname{div}(a\nabla\Phi)+(\nabla a,\nabla\Phi)\geq 0 \text{ in } B_R\setminus B_{(1-\delta)R}\cup B_{(1+\delta)r}\setminus B_r.$$

Since $a[g](v)\leq \frac{\|g\|_{L^1(\mathbb{R}^3)}}{|v|}$, it follows

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} f(v,t)\Phi(v)\,dv &\geq -C_{\delta,r,R}\frac{\|g\|_{L^1(\mathbb{R}^3}}{r}\int_{B_{(1-\delta)R}\setminus B_{(1+\delta)r}} f(v,t)\Phi(v)\,dv\\ &\geq -\frac{\|g\|_{L^1(\mathbb{R}^3)}}{r}C_{\delta,r,R}\int_{\mathbb{R}^3} f(v,t)\Phi(v)\,dv.\end{aligned}$$

This above differential inequality implies

$$\int_{\mathbb{R}^3} f(v,t)\Phi(v)\,dv\geq e^{-\beta T}\int_{\mathbb{R}^3} f_{\text{in}}\Phi(v)\,dv,\ \ \forall\, t<T,$$

where $\beta=C_{r,R,\alpha}\|g\|_{L^1}$. Finally, since

$$\Phi(v)\leq\frac{1}{4}(R-r)^2 \text{ in } B_R\setminus B_r\ ,\ \ \Phi(v)\geq\frac{R^2r^2}{4},$$

we conclude that

$$\int_{B_R\setminus B_r} f(v,t)\,dv\geq\frac{R^2r^2}{(R-r)^4}e^{-\beta T}\int_{B_{R/2}\setminus B_{2r}} f_{\text{in}}(v)\Phi(v)\,dv,\ \ \forall\, t<T.$$

□

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