(i)

Q3 2. Use the Lucas asset-pricing model outlined in the class lecture notes. Modify that model by assuming that there are two trees, type A (for apple) and N (for nectarine). Each yields fruit according to a Markov process. Let the random variable be s and assume that the apple tree has output $a : S \to [0, \bar{a}]$ and the nectarine tree is $n : S \to [0, \bar{n}]$. The household maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$z_{a,t}[p_{a,t} + a_t] + z_{n,t}[p_{n,t} + n_t] \geq c_t + z_{a,t+1}p_{a,t} + z_{n,t+1}p_{n,t} \tag{62}$$

(a) Write down the Bellman equation and derive the first-order conditions and the envelope condition.

Answer The Bellman equation is

$$V(z_{a,t}, z_{n,t}, n_t, a_t) = \max_{\{c_t, z_{a,t+1}, z_{n,t+1}\}} [U(c_t) + \beta E_t V(z_{a,t+1}, z_{n,t+1}, n_{t+1}, a_{t+1})] \tag{63}$$

subject to the budget constraint above. Let λ_t denote the Lagrange multiplier. The first-order conditions and envelope conditions are

$$\begin{aligned} U'(c_t) &= \lambda_t &(64)\\ \lambda_t p_{a,t} &= \beta E_t V_1 &(65)\\ \lambda_t p_{n,t} &= \beta E_t V_2 &(66) \end{aligned}$$

2

$$
\begin{aligned}
V_1 &= \lambda_t[p_{a,t} + a_t] && (67)\\
V_2 &= \lambda_t[p_{n,t} + n_t] && (68)
\end{aligned}
$$

Substitute the envelope conditions into first-order conditions to eliminate the value function derivatives and then eliminate λ_t. The result is a pair of equations

$$
\begin{aligned}
U'(c_t)p_{a,t} &= \beta E_t U'(c_{t+1})[p_{a,t+1} + a_{t+1}] && (69)\\
U'(c_t)p_{n,t} &= \beta E_t U'(c_{t+1})[p_{n,t+1} + n_{t+1}] && (70)
\end{aligned}
$$

(b) Determine the market clearing conditions and substitute into the first-order conditions.
Answer - The market-clearing conditions are

$$
\begin{aligned}
c_t &= a_t + n_t\\
z_{a,t} &= 1\\
z_{n,t} &= 1
\end{aligned}
$$

Substitute into the first-order conditions to obtain

$$
\begin{aligned}
U'(a_t + n_t)p_{a,t} &= \beta E_t U'(a_{t+1} + n_{t+1})[p_{a,t+1} + a_{t+1}] && (71)\\
U'(a_t + n_t)p_{n,t} &= \beta E_t U'(a_{t+1} + n_{t+1})[p_{n,t+1} + n_{t+1}] && (72)
\end{aligned}
$$

(c) Use the covariance decomposition to find the equity risk premium for each type of equity.
Answer - Let $M_{t+1} \equiv \frac{\beta U'(a_{t+1}+n_{t+1})}{U'(a_t+n_t)}$ and define $r_f \equiv [E_t M_{t+1}]^{-1}$ as the risk-free rate of return. Define $R^a_{t+1} = \frac{p_{a,t+1}+a_{t+1}}{p_{a,t}}$ and $R^n_{t+1} = \frac{p_{n,t+1}+n_{t+1}}{p_{n,t}}$ as the rates of return on the equities. Then

$$
\begin{aligned}
E_t R_{a,t+1} - r_{f,t} &= -r_{f,t}\mathrm{Cov}(M_{t+1}, R^a_{t+1})\\
E_t R_{a,t+1} - r_{f,t} &= -r_{f,t}\mathrm{Cov}(M_{t+1}, R^a_{t+1})
\end{aligned}
$$

The intertemporal marginal rate of substitution M is the stochastic discount factor. The covariance of the asset's return with the stochastic discount factor is called the asset's **β-coefficient**.

(d) Suppose that when the apple crop is large, the nectarine crop is small, on average. Hence the dividends on the two equities are negatively correlated. Describe how the two returns will be related using economic intuition.

3

Answer Since the two dividend processes tend to be negatively correlated, consumption tends to be smoother than it would be if the dividend processes were positively correlated. A big drop in apples is likely to be offset by a bumper crop in nectarines so that consumption declines very little. Hence the risk premium for each asset would likely be quite small because, by holding a diversified portfolio, consumption can be quite smooth over time. Compare this to the case where apples and nectarines are positively correlated. Then consumption can't be smoothed over time by holding a diversified portfolio. When the returns to apples and nectarines are high, consumption is also high and hence the stochastic discount factor is low since utility is strictly concave. Hence the covariance for each asset is likely to be large in absolute value (remember it's negative). Hence the risk premiums will be large.