

# Problem Set 4

①

## Answer Key

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Q1.

(a) The state variables are:

$$Z_{m,t}, Z_{h,t}, K_{m,t}, K_{h,t}$$

The control variables are:

$$C_{m,t}, C_{h,t}, K_{m,t+1}, K_{h,t+1}, H_{m,t}, H_{h,t}$$

(b) I use the Lagrange method

$$\begin{aligned} \alpha = & \sum_{t=0}^{\infty} \beta^t E_t \left\{ \frac{1}{e} \log [a C_{m,t}^e + (1-a) C_{h,t}^e] + A \log (1 - h_{m,t} - h_{h,t}) \right. \\ & + \lambda_t \left[ \exp(Z_{m,t}) K_{m,t}^\theta H_{m,t}^{1-\theta} - C_{m,t} - K_{m,t+1} - K_{h,t+1} \right. \\ & \left. \left. + (1-\delta) K_{m,t} + (1-\delta) K_{h,t} \right] \right. \\ & \left. + \mu_t \left[ \exp(Z_{h,t}) K_{h,t}^\eta H_{h,t}^{1-\eta} - C_{h,t} \right] \right\} \end{aligned}$$

The first order conditions are.

(2)

$$\textcircled{1} \quad \frac{a C_{m,t}^{e-1}}{[a C_{m,t}^e + (1-a) C_{h,t}^e]} = \lambda_t$$

$$\textcircled{2} \quad \frac{(1-a) C_{h,t}^{e-1}}{[a C_{m,t}^e + (1-a) C_{h,t}^e]} = \mu_t$$

$$\textcircled{3} \quad \lambda_t = \beta E_t \left\{ \lambda_{t+1} \left[ \theta \exp(Z_{m,t+1}) K_{m,t+1}^{\theta-1} H_{m,t+1}^{1-\theta} + (1-\delta) \right] \right\}$$

$$\textcircled{4} \quad \lambda_t = \beta E_t \left\{ \mu_{t+1} \left[ \eta \exp(Z_{h,t+1}) K_{h,t+1}^{\eta-1} H_{h,t+1}^{1-\eta} \right] + \lambda_{t+1} (1-\delta) \right\}$$

$$\textcircled{5} \quad \frac{A}{1 - H_{m,t} - H_{h,t}} = \lambda_t \exp(Z_{m,t}) (1-\theta) K_{m,t}^{\theta} H_{m,t}^{-\theta}$$

$$\textcircled{6} \quad \frac{A}{1 - H_{m,t} - H_{h,t}} = \mu_t \exp(Z_{h,t}) (1-\eta) K_{h,t}^{\eta} H_{h,t}^{-\eta}$$

Interpretations:

- ① and ② state that the multipliers are respectively the marginal utility of market consumption good and home production good.
- ③ is the intertemporal investment decision on market production capital goods. The marginal cost of investing

in a marginal unit of market production capital (3) good is  $\lambda_t$ , <sup>while</sup> the marginal benefit is next period's marginal product of capital in mkt good production and undepreciated capital stock from this marginal unit.

(4) is a little bit complicated. The marginal cost of investing in one marginal unit of home production capital good is  $\lambda_t$ , as only market goods can be used for the capital stock. The marginal benefit is the marginal product of home production capital goods in units of home production goods and the undepreciated home production capital goods in units of market production goods.

(5) is the labor supply decision on  $H_{m,t}$ , which states that the marginal cost of extra  $H_{m,t}$  is the marginal utility of leisure, and the marginal gain is the marginal utility of consumption coming from marginal product of labor. Interpretation of (6) is similar

(c) when  $e = 1$ , mkt-produced good and home-produced good are perfect substitutes. Individuals are more likely to substitute capital and labor in one sector for the other, leading to higher labor supply elasticity in market activity.

The parameter  $\sigma$  governs their incentives to move production between the two sectors. Lower values of  $\sigma$  entail more frequent divergence between  $Z_m$  and  $Z_n$ , and, hence, more frequent opportunities to specialize over time.

Q2.

(a) State variables:  ~~$K_t$~~   $K_t, A_t$

Control variables:  $C_t(K_{t+1}), Z_t, N_t(L_t)$ .

(b) The extra control variable is capacity utilization decision.

(5)

$$\alpha = \sum_{t=0}^{\infty} \beta^t E_t \left\{ \log C_t + A \log L_t + \lambda_t \left[ A_t (Z_t K_t)^{1-\alpha} N_t^\alpha - C_t - K_{t+1} + [1 - \delta(Z_t)] K_t \right] \right\}$$

w/r.  $C_t$ : (1)  $\frac{1}{C_t} = \lambda_t$

w/r.  $K_{t+1}$ : (2)  $\lambda_t = \beta E_t \left\{ \lambda_{t+1} \left[ A_{t+1} Z_{t+1}^{1-\alpha} K_{t+1}^{-\alpha} N_{t+1}^\alpha + 1 - \delta(Z_{t+1}) \right] \right\}$

w/r.  $Z_t$ : (3)  $(1-\alpha) A_t Z_t^{-\alpha} K_t^{1-\alpha} N_t^\alpha = \delta'(Z_t) K_t$

w/r.  $N_t$ : (4)  $\frac{A}{1-N_t} = \lambda_t \alpha A_t (Z_t K_t)^{1-\alpha} N_t^{\alpha-1}$

(1), (2), (4) are standard like Q 1.

(3) states the decision on the capacity utilization. The marginal benefit is the marginal product from marginal increase in  $Z_t$ , the marginal cost is the marginal acceleration of depreciation of used capital stock.

(c) When there is positive tech shock, (6)  
typically it is worthwhile to raise the capacity utilization even at the cost of higher depreciation. Variable capacity utilization adds a channel thru which output responds to tech shocks.