Problem Set 3

Answer key

Q1. (a)

\[ V(A_t, d_t) = \max \left\{ \frac{C_t^{1-\sigma}}{1-\sigma} + \beta E_t V(A_{t+1}, d_{t+1}) \right\} \]

\[ \{A_{t+1}, C_t\} \]

s.t. \( C_t + p_t A_{t+1} = (p_t + d_t) A_t \)

For w.r.t. \( A_{t+1} \),

\[ C_t^{-\sigma} p_t = \beta E_t V_1(A_{t+1}, d_{t+1}) \]

Env:

\[ V_1(A_t, d_t) = C_t^{-\sigma} (p_t + d_t) \]

End.

\[ \Rightarrow \left[ C_t^{-\sigma} p_t = \beta E_t \left[ C_{t+1}^{-\sigma} (p_{t+1} + d_{t+1}) \right] \right] \]

Market clearing conditions:

\[ \begin{align*}
C_t &= d_t \\
A_t &= 1
\end{align*} \]
(b)

\[ P_t = \beta \ E_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^\theta \ (P_{t+1} + d_{t+1}) \right] \]

Define \( P_i \) to be the price of the asset when \( d_t = d_i \).

\[ \Rightarrow P_i = \beta \left[ \theta \left( \frac{d_i}{d_1} \right) (P_i + d_i) + (1-\theta) \left( \frac{d_2}{d_1} \right) (P_2 + d_2) \right] \]

\[ P_2 = \beta \left[ (1-\theta) \left( \frac{d_2}{d_2} \right) (P_2 + d_2) + \theta \left( \frac{d_2}{d_2} \right) (P_2 + d_2) \right] \]

(c) In the state \( d_t = d_1 \),

\[ E \left[ \frac{P_{t+1} + d_{t+1}}{P_t} \mid d_t = d_1 \right] = \frac{E(\ P_{t+1} + d_{t+1}) \mid d_t = d_1}{P_i} \]

\[ = \frac{1}{P_i} \left[ \theta (P_i + d_i) + (1-\theta) (P_2 + d_2) \right] \]

Accordingly,
\[ E\left[ \frac{P_{t+1} + d_{t+1}}{P_t} \left| d_t = d_2 \right. \right] = \frac{E\left[ (P_{t+1} + d_{t+1}) \left| d_t = d_2 \right. \right]}{p_2} \]

\[ = \frac{1}{p_2} \left[ (1-\theta)(P_1 + d_1) + \theta (P_2 + d_2) \right] \]

(d) Since the conditional expected gross asset return is a function of the state variable \( d_t \), so the unconditional Expectation

\[ E \left( \frac{P_{t+1} + d_{t+1}}{P_t} \right) = E \left( \frac{P_{t+1} + d_{t+1}}{P_t} \left| d_t = d_1 \right. \right) \text{prob}(d_t = d_1) \]

\[ + E \left( \frac{P_{t+1} + d_{t+1}}{P_t} \left| d_t = d_2 \right. \right) \text{prob}(d_t = d_2) \]

where \( \text{prob}(d_t = d_i) \) is the unconditional prob of being in state \( i \).
Q2. (a) $P_{nt} = E_t \left[ \beta^n \frac{U'(C_{tn})}{U'(C_t)} \right]$. Since there is only one asset, $C_t = d_t$

In equilibrium,

$$P_{nt} = E_t \left( \beta^n \frac{U(d_{tn})}{U(d_t)} \right)$$

(b) $\log(P_{nt}) = \log \left\{ E_t \left[ \beta^n \left( \frac{d_{tn}}{d_t} \right)^{-\gamma} \right] \right\}$

Given the dividend process, $-\gamma (\log d_{tn} - \log d_t)$ is conditionally normal,

$$= n \log \beta + \log \left\{ E_t \left[ \exp \left( -\frac{\gamma^2}{2} \text{Var}(\log d_{tn} - \log d_t) \right) \right] \right\}$$

$$= n \log \beta + \log \left\{ \exp \left( \frac{-\gamma^2}{2} \text{Var}(\log d_{tn} - \log d_t) \right) \right\}$$

$$= n \log \beta - \gamma E_t (\log d_{tn} - \log d_t) + \frac{\gamma^2}{2} \text{Var}(\log d_{tn} - \log d_t)$$

From the dividend process,

$$\log (d_{tn}) - \log (d_t) = \delta + \epsilon_{nt}$$

$$\log (d_{tn}) - \log (d_t) = \log (d_{tn}) - \log (d_{n-1}) + \log (d_{n-1}) - \log (d_t)$$

$$= \delta + \epsilon_{n-1} + \delta + \epsilon_{n-2} + \delta + \epsilon_{n-1}$$

$$= 2 \delta + \epsilon_{n-1} + \epsilon_{n-2} + \epsilon_{n-1}$$

Continue the derivation,

$$\Rightarrow \log (d_{tn}) - \log d_t = n \delta + \epsilon_{tn} + \epsilon_{tn-1} + \cdots + \epsilon_{t+1}$$

$$\Rightarrow E_t [\log (d_{tn}) - \log d_t] = n \delta$$

$$\text{Var}_t [\log (d_{tn}) - \log d_t] = n \sigma^2$$

due to white noise assumption of $\epsilon_t$.

$$\Rightarrow r_{nt} = -\frac{\log(P_{nt})}{n} = -\log \beta + \delta \gamma - \frac{\gamma^2 \sigma^2}{2}$$

In this particular case, where $\log d_t$ is a random walk with a drift, $r_{nt}$ does not depend upon $n$. Thus the term structure of
Note that the key reason for the flat yield curve in (b) above is that \( E_t(\log d_{t+n} - \log d_{t+1}) = \theta \), that is, the expected consumption growth is the same from one period to the next, we want to change this specification to make an upward or downward sloping yield curve possible. One specification would be:

\[
\log(d_t) = \theta + \log(d_{t-1}) + e_t + \theta e_{t-1}, \quad \text{i.e. MA(1)}
\]

the consumption growth, \( \log d_t - \log d_{t+1} \) follows an \( MA(1) \) process.

\[
\Rightarrow \log d_{t+n} - \log d_t = n\theta + e_{t+n} + (1+\theta)e_{t+n-1} + \cdots + (1+\theta)e_{t+n} + \theta e_t
\]

\[
E_t(\log d_{t+n} - \log d_t) = n\theta + \theta e_t
\]

\[
\text{Var}_t(\log d_{t+n} - \log d_t) = \sigma^2 + (n-1)(1+\theta)^2 \sigma^2 = n\sigma^2 + (n-1)(2\theta + \theta^2)\sigma^2
\]

from Derivation on page 2.

\[
\eta_{nt} = -\log(\theta) + \frac{\gamma E_t(\log d_{t+n} - \log d_t)}{n} - \frac{\gamma^2}{2n^2} \text{Var}_t(\log d_{t+n} - \log d_t)
\]

\[
= -\log(\theta) + \gamma \theta - \frac{\gamma^2}{2n} \sigma^2 + \frac{\gamma \theta}{n} e_t - \frac{(n-1)\sigma^2 (2\theta+\theta^2)}{2n}
\]

Define \( f(n, e_t) = \frac{\theta}{n} e_t - \frac{(n-1)\sigma^2 (2\theta+\theta^2)}{2n} \).

when \( e_t > 0 \), \[ f(n, e_t) \frac{\partial}{\partial n} < 0 \), the yield curve is downward sloping for sure.
To write it out fully, 

\[ \frac{\partial f(n, e_t)}{\partial n} = -\frac{\gamma}{n^2} e_t + \frac{1}{n^2} \frac{\sigma^2}{2} (2\theta + \theta^2) \delta^2 \]

When 

\[ -\frac{\gamma}{n^2} e_t + \frac{1}{n^2} \frac{\sigma^2}{2} (2\theta + \theta^2) \delta^2 > 0, \]

the yield curve is upward sloping, i.e.,

when 

\[ e_t < -\frac{\gamma (2\theta + \theta^2) \delta^2}{2} < 0. \]

The yield curve is upward sloping when \( e_t \) is negative and smaller than \( -\frac{\gamma (2\theta + \theta^2) \delta^2}{2} \).

Intuition:

When \( e_t > 0, \) \( e_t (\log d_{tt+1} - \log d_t) > E_t (\log d_{tt+1} - \log d_{tt+1}) \), for \( n > 1 \), after the positive shock from \( e_t \), the economy grows down in the following periods. The agents would be willing to save even at lower future short term interest rates.

When \( e_t \) is negative, the current state economy is worse than the periods ahead. The agents would like to borrow from future and drive up the future short term interest rates. However, \( e_t \) must be small enough to counter the increase in consumption growth uncertainty which pushes the term structure downward.
B. The consumer maximizes

$$E_0 \sum_{t=0}^{\infty} \frac{u(C_t)}{(1+r)^t}, \text{ where } u(C_t) = \frac{C_t^{1-\theta}}{1-\theta}, \theta > 0$$

s.t. \( \frac{b+t}{1+r} + C_t \leq Y_t + b_t \)

(a) The Euler equation is

$$\frac{(1+r)}{(1+r)} E_t [u'(C_{t+1})] = u'(C_t)$$

$$\Rightarrow \frac{(1+r)}{(1+r)} E_t (C_{t+1}^{-\theta}) = C_t^{-\theta}$$

(b) The Euler equation can be rewritten as

$$\frac{(1+r)}{(1+r)} E_t \left[ \exp(-\theta \ln C_{t+1}) \right] = \exp(-\theta \ln C_t)$$

$$\Rightarrow \frac{(1+r)}{(1+r)} \exp \left[ -\theta E_t (u(C_{t+1}) + \frac{\theta^2}{2} \delta^2) \right] = \exp(-\theta \ln C_t)$$
\[ \Rightarrow \ln(1+r) - \ln(1+P) - \Theta E_t(\ln C_{tt}) + \frac{\Theta}{2} \delta^2 = -\Theta \ln C_t \]

(c) If \( r \) and \( \delta^2 \) are constant over time,

\[ E_t \ln C_{tt} = -\frac{1}{\Theta} \left[ \ln(1+P) - \ln(1+r) - \frac{\Theta}{2} \delta^2 \right] + \ln C_t \]

\[ \Rightarrow \ln C_{tt} = -\frac{1}{\Theta} \left[ \ln(1+P) - \ln(1+r) - \frac{\Theta}{2} \delta^2 \right] + \ln C_t + u_{tt} \]

(d) \[ E_t \ln C_{tt} - \ln C_t = -\frac{1}{\Theta} \left[ \ln(1+P) - \ln(1+r) - \frac{\Theta}{2} \delta^2 \right] \]

\[ \frac{\partial (E_t \ln C_{tt} - \ln C_t)}{\partial \delta^2} = \frac{1}{\Theta (1+r)} > 0 \]

As \( r \uparrow \), the consumer is more willing to delay consumption to the next period. As a result, expected consumption growth goes up.

\[ \frac{\partial (E_t \ln C_{tt} - \ln C_t)}{\partial \delta^2} = \frac{\Theta}{2} > 0 \]

As \( \delta^2 \uparrow \), the consumer engages in precautionary saving in light of higher uncertainty. As a result, current consumption is suppressed and expected consumption growth goes up.
Q4.

(a) \[ A_{t+1} = (1+r)(A_t - C_t + W_t) \]

\[ \Rightarrow A_t = \frac{1}{(1+r)} A_{t+1} - W_t + C_t \]

\[ \left[ 1 - \frac{1}{1+r} F \right] A_t = C_t - W_t \]

\[ A_t = \frac{1}{1 - \frac{1}{1+r} F} \left( C_t - W_t \right) \]

\[ \Rightarrow A_t = \sum_{j=0}^{\infty} \left( \frac{1}{1+r} \right)^j \left[ C_{t+j} - W_{t+j} \right] \]

\[ \Rightarrow \sum_{j=0}^{\infty} \left\{ \left( \frac{1}{1+r} \right)^j C_{t+j} \right\} = A_t + \sum_{j=0}^{\infty} \left\{ \left( \frac{1}{1+r} \right)^j W_{t+j} \right\} \]

(b) \[ V(A_t, W_t) = \max \left\{ \frac{C_t^{1-r}}{1-r} + \beta V(A_{t+1}, W_{t+1}) \right\} \]

s.t. \[ A_{t+1} = (1+r)(A_t - C_t + W_t) \]
(c.) 

\[ \frac{C_t}{1 + r} = \beta V_1(A_{t+1}, W_{t+1}) \]

E.m. \[ V_1(A_t, W_t) = C_t^{-\gamma} \]

\[ \Rightarrow C_t^{-\gamma} = \beta (1 + r) C_{t+1}^{-\gamma} \Rightarrow \left( \frac{C_{t+1}}{C_t} \right) = \beta (1 + r) \]

(c.d) The Euler equation implies:

\[ \frac{C_{t+1}}{C_t} = \left[ \beta (1 + r) \right] ^{\frac{1}{\gamma}} \implies C_{t+1} = \left[ \beta (1 + r) \right] ^{\frac{1}{\gamma}} C_t \]

Substitute into the lifetime budget:

\[ \Rightarrow \sum_{j=0}^{\infty} \left\{ \left( \frac{1}{1 + r} \right) C_0 \left[ \beta (1 + r) \right] ^{\frac{j}{\gamma}} \right\} = A_t + \sum_{j=0}^{\infty} \left\{ \left( \frac{1}{1 + r} \right) W_{t+j} \right\} \]
\( V(A_t, W_t) = \max \left\{ \frac{C_t^{1-\gamma}}{C_t} + \beta V(A_{t+1}, W_{t+1}) + \mu_t A_{t+1} \right\} \)

\[
\text{Fac:} \quad \frac{C_t^{-\gamma}}{1+r} = \beta V_1(A_{t+1}, W_{t+1}) + \mu_t
\]

\[
\text{Env:} \quad V_1(A_t, W_t) = C_t^{-\gamma}
\]

\[
\Rightarrow \quad C_t^{-\gamma} = \beta (1+r) C_{t+1}^{-\gamma} + \mu_t (1+r)
\]

When the constraint is binding (\( \mu_t > 0 \)),

\[
C_t^{-\gamma} > \beta (1+r) C_{t+1}^{-\gamma}
\]

When the constraint is not binding (\( \mu_t = 0 \)),

\[
C_t^{-\gamma} = \beta (1+r) C_{t+1}^{-\gamma}
\]
Term structure budget constraint.

\[ C_t + P_t A_{t+1} + \sum_{n=1}^{N} P_{n,t} B_{n,t+1} \]

\[ = (P_t + d_t) A_t + B_{1,t} + \sum_{n=1}^{N-1} P_{n,t} B_{n+1,t} \]

where \( \{B_{n+1,t}\} \) are determined at period \( t \),

where \( \{B_{n+1,t}\} \) are determined at period \( t-1 \), thus \( B_{n+1,t} \) has 
\( n \) periods to maturity at period \( t \).

State variables: \( A_t, \{B_{n,t}\}_{n=1}^{N}, d_t \)

Control variables: \( \{B_{n,t+1}\}_{n=1}^{N} \)