**Problem 1** The economy is populated by a continuum of identical households of unit one. The representative household maximizes

\[ E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \]

subject to the following budget constraint,

\[ c_t + p_t a_{t+1} = (p_t + d_t) a_t, \]

where \( p_t \) denotes the price of a share of the asset in terms of the consumption good, \( a_{t+1} \) denotes the quantity of shares held from period \( t \) to \( t+1 \), and \( d_t \) represents dividends. Suppose that \( d_t \) takes on only two values, \( \{d_1, d_2\} \) with \( d_1 > d_2 \). Further, suppose that \( d_t \) follows a two-state Markov chain, that is

\[ \Pr[d_{t+1} = d_j | d_t = d_i] = \pi_{ij}. \]

We will assume that \( \pi_{ii} = \theta \), for \( i = 1, 2 \).

(a) Formulate the Bellman equation, Derive the Euler equation(s) and state the market-clearing conditions.

(b) Derive the two equations which can be used to solve \( p_t \) respectively when \( d_t = d_h, d_l \).

(c) Derive the expected gross return on the asset conditional on the information at period \( t \).

(d) Describe how you compute the unconditional expectation of the gross asset return.

**Problem 2** The term structure of interest rates. *(This is a simplified version of Campbell QJE 1986)*

Consider the Lucas asset-pricing model, with only one asset. Assume that the dividend on the asset follows

\[ \log(d_t) = g + \log(d_{t-1}) + e_t, \]
where $e_t$ is normally distributed white noise, with mean zero and variance $\sigma^2$. Assume that utility is CRRA, with a coefficient of relative risk aversion of $\gamma$.

(a) Consider a $n$-period pure discount bond at time $t$, a bond that pays one unit of the good at time $t+n$ and has price $p_{n,t}$. Derive the equilibrium values of $p_{n,t}$.

(b) Define the yield to maturity on a $n$-period pure discount bond as $r_{n,t}$, such that

$$r_{n,t} = -\frac{\log(p_{n,t})}{n}.$$ 

Characterize the term structure of yields to maturity at time $t$. Is it upward sloping or downward sloping? Explain.

(c) How would we have to change the model to explain why the term structure is sometimes upward sloping and sometimes downward sloping?

**Problem 3** Suppose instantaneous utility is of the constant relative risk aversion form, $u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta}, \theta > 0$. Assume that the real interest rate, $r$, is constant but not necessarily equal to the subjective discount rate, $\rho$.

(a) Find the Euler equation relating $C_t$ to expectations concerning $C_{t+1}$.

(b) Suppose that the log of income is distributed normally, and that as a result the log of $C_{t+1}$ is distributed normally; let $\sigma^2$ denote its variance conditional on information available at time $t$. Rewrite the expression in part (a) in terms of $\ln C_t, E_t \ln C_{t+1}, \sigma^2,$ and the parameters $r, \rho,$ and $\theta$.

(c) Show that if $r$ and $\sigma^2$ are constant over time, the result in part (b) implies that the log of consumption follows a random walk with drift:

$$\ln C_{t+1} = \alpha + \ln C_t + e_{t+1},$$

where $e$ is white noise.

(d) How do changes in each of $r$ and $\sigma^2$ affect expected consumption growth, $E_t [\ln C_{t+1} - \ln C_t]$? Interpret the effect of $\sigma^2$ on expected consumption growth in light of the presence of precautionary saving.

**Problem 4** Consider a consumer with initial assets $A_0$ and preferences

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma}, \gamma > 0$$
where $0 < \beta < 1, c_t$ is consumption. The consumer’s budget constraint is

$$A_{t+1} = (1 + r)(A_t - c_t + w_t), \quad (1)$$

for $t = 0, 1, 2, ..., $ where $r$ is the one-period interest rate (assumed constant over time) and $w_t$ is income in period $t$, where income is exogenous. We also assume the no-Ponzi-scheme condition

$$\lim_{t \to \infty} \frac{A_t}{(1 + r)^t} = 0. \quad (2)$$

(a) Derive the lifetime budget constraint.

(b) Formulate the Bellman equation for the consumer’s problem.

(c) Find the first order conditions for each control variable respectively and derive the Envelop condition.

(d) Without assuming $\beta (1 + r) = 1$, derive the solution for $c_0$, and the entire optimal consumption path.

(e) Derive the Euler equation when there is a no-borrowing constraint $A_{t+1} \geq 0$. Interpret the economic intuition behind this Euler equation.