Answer Key to Problem Set 3

Econ 306

Instructor: Chao Wei

Q1.

$$1 = \beta E_t\left[\frac{u'(C_{t+1})}{u'(C_t)}(1+R_{it})\right]$$

$$\Rightarrow 1 = \beta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(1+R_{it})\right]$$

$$\Rightarrow 1 = \beta E_t\left\{\exp\left[-\gamma(\log C_{t+1} - \log C_t) + \log(1+R_{it})\right]\right\}$$

$$\Rightarrow 1 = \beta \exp\left\{-\gamma E_t(\log C_{t+1} - \log C_t) + E_t \log(1+R_{it}) + \frac{1}{2}Var_t\left[-\gamma(\log C_{t+1} - \log C_t) + \log(1+R_{it})\right]\right\}$$

$$\Rightarrow 0 = \log\beta - \gamma g + x + \frac{1}{2}(\gamma^2 V_{11} - 2\gamma V_{12} + V_{22}),$$

where V_{ij} is the (i,j)th element of V.

$$\Rightarrow x = -\log\beta + \gamma g - \frac{1}{2}(\gamma^2 V_{11} - 2\gamma V_{12} + V_{22})$$

Q2.

(a) $$P_{n,t} = E_t\left(\beta^n \frac{u'(C_{t+n})}{u'(C_t)}\right)$$

Since $\exists$ only one asset, $C_t = d_t$

In equilibrium.

$$p_{n,t} = E_t\left[\beta^n \frac{u'(d_{t+n})}{u'(d_t)}\right]$$

(b) $$\log(p_{n,t}) = \log\left\{E_t\left[\beta^n\left(\frac{d_{t+n}}{d_t}\right)^{-\gamma}\right]\right\}$$

$$= n\log\beta + \log\left\{E_t\left[\exp(-\gamma[\log d_{t+n} - \log d_t])\right]\right\}$$

Given the dividend pass, $-\gamma(\log d_{t+n} - \log d_t)$ is conditionally normal,

$$= n\log\beta + \log\left\{\exp\left(-\gamma E_t(\log d_{t+n} - \log d_t) + \frac{\gamma^2}{2} Var_t(\log d_{t+n} - \log d_t)\right)\right\}$$

$$= n\log\beta - \gamma E_t(\log d_{t+n} - \log d_t) + \frac{\gamma^2}{2} Var_t(\log d_{t+n} - \log d_t)$$

From the dividend process.

$$\log(d_{t+1}) - \log(d_t) = g + e_{t+1}$$

$$\log(d_{t+2}) - \log(d_t) = \log(d_{t+2}) - \log(d_{t+1}) + \log(d_{t+1}) - \log d_t$$

$$= g + e_{t+2} + g + e_{t+1}$$

$$= 2g + e_{t+2} + e_{t+1}$$

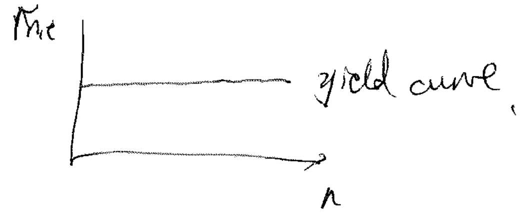
continue the derivation $\Rightarrow \log(d_{t+n}) - \log d_t = ng + e_{t+n} + e_{t+n-1} + \cdots + e_{t+1}$

$\Rightarrow$ $E_t[\log(d_{t+n}) - \log d_t] = ng$

$Var_t[\log(d_{t+n}) - \log d_t] = n\sigma^2$ due to white noise assumption of e_t.

$$\Rightarrow r_{n,t} = -\frac{\log(p_{n,t})}{n} = -\log\beta + g\gamma - \frac{\gamma^2\sigma^2}{2}$$

In this particular case, where $\log(d_t)$ is a random walk with a drift, $r_{n,t}$ does not depend upon n. Thus the term structure of yields to maturity at time t can be described as a flat yield curve.



(c) Note that the key reason for the flat yield curve in (b) above is that $E_t(\log d_{t+n} - \log d_{t+n-1}) = g$, that is, the expected consumption growth is the same from one period to the next, we want to change this specification to make an upward or downward sloping yield curve possible

One specification would be

assume $\theta > 0$

$$\log(d_t) = g + \log(d_{t-1}) + e_t + \theta e_{t-1}, \text{ i.e.}$$

the consumption growth, $\log d_t - \log d_{t-1}$ follows an MA(1) process.

$$\Rightarrow \log d_{t+n} - \log d_t = ng + e_{t+n} + (1+\theta)e_{t+n-1} + \cdots + (1+\theta)e_{t+1} + \theta e_t$$

$$E_t(\log d_{t+n} - \log d_t) = ng + \theta e_t$$

$$Var_t(\log d_{t+n} - \log d_t) = \sigma^2 + (n-1)(1+\theta)^2\sigma^2 = n\sigma^2 + (n-1)(2\theta+\theta^2)\sigma^2$$

From Derivation on page ②.

$$r_{n,t} = -\log\beta + \frac{\gamma E_t(\log d_{t+n} - \log d_t)}{n} - \frac{\gamma^2}{2}\frac{Var_t(\log d_{t+n} - \log d_t)}{n}$$

$$= -\log\beta + \gamma g - \frac{\gamma^2}{2}\sigma^2 + \frac{\gamma\theta}{n}e_t - \frac{(n-1)}{n}\frac{\gamma^2(2\theta+\theta^2)\sigma^2}{2}$$

Define $f(n, e_t) = \frac{\theta}{n}e_t - \frac{(n-1)}{n}\frac{\gamma^2}{2}(2\theta+\theta^2)\sigma^2$,

when $e_t > 0$, $\partial f(n, e_t)/\partial n < 0$, the yield curve is downward sloping for sure.

3.5

~~when~~ To write it out fully.

$$\frac{\partial f(n, e_t)}{\partial n} = -\frac{\gamma\theta}{n^2} e_t + \frac{1}{n^2} \frac{\gamma^2}{2}(2\theta + \theta^2)\sigma^2$$

~~when e_t~~ when $-\frac{\gamma\theta}{n^2} e_t + \frac{1}{n^2} \frac{\gamma^2}{2}(2\theta + \theta^2)\sigma^2 > 0$,

the yield curve is upward sloping, i.e.

when $e_t < -\frac{\gamma(2+\theta)\sigma^2}{2} < 0$.

The yield curve is upward sloping when e_t is negative and smaller than $\frac{\gamma(2+\theta)\sigma^2}{2}$.

Intuition:

when $e_t > 0$, $E_t(\log d_{t+1} - \log d_t) > E_t(\log d_{t+n} - \log d_{t+n-1})$. for $n > 1$, after the positive shock from e_t, the economy goes down in the following periods. The agents would be willinging to save even at lower future short term interest rates.

when e_t is negative, the current ~~st~~ economy is worse than periods ahead. the agents would like to borrow from the future and drive up the future short term interest rates. However, e_t must be small enough to counter the increase in consumption growth uncertainty which pushes the term structure downward.

3. The consumer maximizes

$$E_0 \sum_{t=0}^{\infty} u(C_t)/(1+\rho)^t, \quad \text{where } u(C_t) = \frac{C_t^{1-\theta}}{1-\theta}, \quad \theta > 0$$

s.t.
$$\frac{b_{t+1}}{1+r} + C_t \leq y_t + b_t$$

(a) The Euler equation is

$$\frac{(1+r)}{(1+\rho)} E_t\left[u'(C_{t+1})\right] = u'(C_t)$$

$$\Rightarrow \frac{(1+r)}{(1+\rho)} E_t(C_{t+1}^{-\theta}) = C_t^{-\theta}$$

(b) The Euler equation can be rewritten as

$$\frac{(1+r)}{(1+\rho)} E_t\left[\exp(-\theta \ln C_{t+1})\right] = \exp(-\theta \ln C_t)$$

$$\Rightarrow \frac{(1+r)}{(1+\rho)} \exp\left[-\theta E_t(\ln C_{t+1}) + \frac{\theta^2}{2}\sigma^2\right] = \exp(-\theta \ln C_t)$$

$$\Rightarrow \ln(1+r) - \ln(1+\rho) - \theta E_t(\ln C_{t+1}) + \frac{\theta^2}{2}\sigma^2 = -\theta \ln C_t$$

(c) If r and σ^2 are constant over time,

$$E_t \ln C_{t+1} = -\frac{1}{\theta}\left[\ln(1+\rho) - \ln(1+r) - \frac{\theta^2}{2}\sigma^2\right] + \ln C_t$$

$$\Rightarrow \ln C_{t+1} = -\frac{1}{\theta}\left[\ln(1+\rho) - \ln(1+r) - \frac{\theta^2}{2}\sigma^2\right] + \ln C_t + u_{t+1}$$

(d) $$E_t \ln C_{t+1} - \ln C_t = -\frac{1}{\theta}\left[\ln(1+\rho) - \ln(1+r) - \frac{\theta^2}{2}\sigma^2\right]$$

$$\frac{\partial(E_t \ln C_{t+1} - \ln C_t)}{\partial r} = \frac{1}{\theta(1+r)} > 0$$

As $r \uparrow$, the consumer is more willinging to delay consumption to the next period. As a result, expected consumption growth goes up.

$$\frac{\partial(E_t \ln C_{t+1} - \ln C_t)}{\partial \sigma^2} = \frac{\theta}{2} > 0$$

As $\sigma^2 \uparrow$, the consumer engages in precautionary saving in light of higher uncertainty. As a result, current consumption is suppressed and expected consumption growth goes up.