

# Notes on Macroeconomic Theory II

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# 1 Deterministic Dynamic Programming

Below I describe a typical dynamic programming problem.

## 1.1 Equivalence between Two Problems<sup>1</sup>

The problem is:

$$\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \quad (1)$$

subject to

$$\begin{aligned} 0 &\leq k_{t+1} \leq (1 - \delta) k_t + F(k_t, n_t) - c_t, \\ k_0 &> 0 \end{aligned} \quad (2)$$

Suppose the problem has been solved for all possible values of  $k_0$ . Then we could define a function  $v : \mathbf{R}_+ \rightarrow \mathbf{R}$  by taking  $v(k_0)$  to be the value of the maximized objective function (1), for each  $k_0 > 0$ .

A function of this sort is called a *value function*, and the arguments of this function are called state variables.

With  $v$  so defined,  $v(k_1)$  would give the value of the utility from period 1 on that could be obtained with a beginning-of-period capital stock  $k_1$ , and  $\beta v(k_1)$  would be the value of this utility discounted back to period 0.

Then in terms of this value function  $v$ , the planner's problem in period 0 would be

$$\max_{c_0, k_1} [u(c_0) + \beta v(k_1)] \quad (3)$$

such that

$$\begin{aligned} c_0 + k_1 &\leq (1 - \delta) k_0 + F(k_0, n_0) \\ c_0, k_1 &\geq 0, k_0 > 0 \text{ given} \end{aligned}$$

If the function  $v$  were known, we could use (3) to define a function  $g : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  as follows: for each  $k_0 \geq 0$ , let  $k_1 = g(k_0)$  and  $c_0 = (1 - \delta) k_0 +$

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<sup>1</sup>The notes in this section are adapted from Stokey and Lucas (1989) Section 2.1, and Ljungqvist and Sargent (2004) Chapter 3.

$F(k_0, n_0) - g(k_0)$  be the values that attain the maximum in (3). Here  $g(\cdot)$  is called *the policy function* and consumption and capital stock in the next period are *control variables*.

We do not at this point “know”  $v$ , but we have defined it as the maximized objective function for the problem in (1). Thus if solving (3) provides the solution for that problem, then  $v(k_0)$  must be the maximized objective function for (3) as well. That is,  $v$  must satisfy

$$v(k_0) = \max_{0 \leq k_1 \leq (1-\delta)k_0 + F(k_0, n_0)} \{u[(1-\delta)k_0 + F(k_0, n_0) - k_1] + \beta v(k_1)\}.$$

When the problem is looked at in this recursive way, the time subscripts have become a nuisance: we do not care about what the date is. We can rewrite the problem facing a planner with current capital stock  $k$  as

$$v(k) = \max_{0 \leq k' \leq f(k)} \{u[f(k) - k'] + \beta v(k')\}. \quad (4)$$

This one equation in the unknown function  $v$  is called *a functional equation*, and as shown in Stokey and Lucas (1989), it is a very tractable mathematical object. The study of dynamic optimization problems through the analysis of such functional equations is called *dynamic programming*.

If we knew that the function  $v$  was differentiable and that the maximizing value of  $k'$ , call it  $g(k)$ , was interior, then the first-order and envelope conditions for (4) would be

$$\begin{aligned} u'[f(k) - g(k)] &= \beta v'[g(k)], \text{ and} \\ v'(k) &= f'(k) u'[f(k) - g(k)], \end{aligned}$$

respectively. Note that in deriving the above equations, we used the envelope condition that when  $k'$  is valued at the maximized policy function  $g(k)$ , the derivative  $u(c) + \beta v(k')$  with respect to  $k'$  is 0.

The first of these conditions equates the marginal utility of consuming current output to the marginal utility of allocating it to capital stock and enjoying augmented consumption next period.

The second condition states that the marginal value of current capital, in terms of total discounted utility, is given by the marginal utility of using the capital in current production and allocating its return to current consumption.

Under conditions specified in details in Stokey and Lucas (1989),

1. The functional equation (4) has a unique strictly concave solution.
2. The solution is approached in the limit as  $j \rightarrow \infty$  by iterations on

$$v_{j+1}(k) = \max_{0 \leq k' \leq f(k)} \{u[f(k) - k'] + \beta v_j(k')\},$$

starting from any bounded and continuous initial  $v_0$ .

3. There is a unique and time-invariant optimal policy of the form  $k' = g(k)$ , where  $g$  is chosen to maximize the right side of equation (4).

## 1.2 Computational Methods

We introduce two main types of computational methods:

1. Value function iteration<sup>2</sup>.
2. Guess and verify.

## 1.3 Examples

**Example 1** *The problem described in (1), with  $u(c) = \ln(c)$ ,  $Y = Ak^\alpha$ , and  $\delta = 1$ .*

(1) *Conduct the first two steps of value function iteration by hand;*

(2) *Guess and verify the solution (Hint:  $v(k) = E + F \ln k$ ;*

(3) *Derive the first-order condition, envelop condition and in the end, Euler equation for this problem.*

**Example 2** *The problem described in (1), with  $u(c) = c - \theta c^2$ ,  $\theta > 0$ . Assume that  $c$  is always in the range where  $u'(c)$  is positive. Output is linear in capital,  $y = Ak$ , and  $\delta = 0$ .*

(1) *Find the Euler equation relating  $c_t$  and expectations of  $c_{t+1}$ ;*

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<sup>2</sup>FYI, read Appendix A of Ljungqvist and Sargent (2004) and Stokey and Lucas (1989) to learn what it means for a sequence of functions to converge.

(2) Guess that consumption takes the form  $c_t = \alpha + Fk_t$ . Given this guess, what is  $k_{t+1}$  as a function of  $k_t$ ?

(3) What values must the parameters  $\alpha$  and  $F$  have for the first order condition in part (1) to be satisfied for all values of capital stock?

**Example 3** *Cake-eating problem where  $u(c) = \ln(c)$ ,  $Y = k$ . (Hint: The Euler equation gives the transition rule of consumption. Equilibrium requires that consumption adds up to be the size of the cake.)*

**Example 4** *Give hint on multiple control variables, such as labor.*

Additional readings on examples: Williamson, Chapter 4; Stokey and Lucas, Chapter 5.

## 1.4 Competitive Equilibrium

As described in Williamson's notes.

## 2 Stochastic Dynamic Programming

We will focus on the social planner's problem in this setting, leaving the descriptions of competitive equilibrium to the chapter on asset pricing. We still work with the problem described in (1), but now uncertainty affects the technology in a specific way.

Assume that the households in this economy rank stochastic consumption sequences according to the expected utility they deliver, where their underlying (common) utility function takes the same additively separable form as before:

$$E[u(c_0, c_1, \dots)] = E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]. \quad (5)$$

Here  $E(\cdot)$  denotes expected value with respect to the probability distribution of the random variables  $\{c_t\}_{t=0}^{\infty}$ .

Assume that output is given by  $y_t = z_t f(k_t)$  where  $\{z_t\}$  is a sequence of independently and identically distributed random variables, and  $f(k) = F(k, 1) + (1 - \delta)k$ . The feasibility constraints for the economy are then

$$k_{t+1} + c_t \leq z_t f(k_t), c_t, k_{t+1} \geq 0, \text{ all } t, \text{ all } \{z_t\}. \quad (6)$$

Now consider the problem facing a benevolent social planner in this stochastic environment. His objective function is to maximize the objective function in (5) subject to the constraints in (6). Before proceeding, we need to be clear about the timing of information and decisions, about the objects of choice for the planner, and about the distribution of the random variables  $\{c_t\}_{t=0}^{\infty}$ .

At the beginning of period  $t$ , the state of the economy is characterized by  $k_t$  and the history of exogenous shocks  $\{z_t\}_{t=0}^{\infty}$ . Since  $z_t$  is an i.i.d. random variable, the state variables for our particular problem are  $\{k_t, z_t\}$ . The feasible set for the planner is the set of pairs  $\{c_0, k_1\}$  and sequences of functions  $\{[c_t(\cdot), k_{t+1}(\cdot)]\}_{t=1}^{\infty}$  that satisfy (5) for all periods and all realizations of shocks. We call these sequence of functions as *contingency plans*.

**Example 5** Consider the finite horizon case where  $T = 2$ . Suppose that there are only two values  $z_t$  can take. Consider the cases when  $z_t$  is an i.i.d or Markov process.

(1) At the period  $T = 2$ , what is the optimal choice of  $c_2$ ?

(2) Write the Bellman equation at the period  $T = 1$  and derive the Euler equation. Do the same for the period  $T = 0$ .

**Example 6** The problem described in (1), with  $u(c) = \ln(c)$ ,  $Y = zk^\alpha$ , and  $\delta = 1$ . Assume that  $E[\ln z] = \mu$ .

(1) Guess and verify the solution (Hint:  $v(k, z) = E + F \ln k + F \ln z$ ;

(2) Derive the first-order condition, envelop condition and in the end, Euler equation for this problem.

(3) Will  $k_t$  converge to a steady state value as in the deterministic model? Can anything be said about the behavior of the sequence  $\{k_t\}$ ?

The solution to the above example is called *stochastic difference equation*. Under suitable conditions, the solution can characterize an invariant distribution for the transition distribution function, which are useful for computing moments of random variables.

**Example 7** Search Model. P172 of LS

(1) Effect of unemployment compensation, mean-preserving spreads;

(2) Allowing quits

### 3 Consumption<sup>3</sup>

Consumption theory typically treats asset prices as being exogenous and determines optimal consumption-savings decision for a consumer. The main feature of the data that consumption theory aims to explain is that aggregate consumption is smooth, relative to aggregate income.

#### 3.1 Consumption Behavior Under Certainty

A representative household maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{7}$$

subject to

$$c_t + R^{-1}b_{t+1} \leq y_t + b_t, \text{ where } b_0 \text{ is given.}$$

Assumptions:

- $u(\cdot)$  is a strictly increasing, strictly concave, twice-differentiable one-period utility function.
- Inada condition:  $\lim_{c \rightarrow 0} u'(c) = +\infty$ ;
- No Uncertainty;
- $\{y_t\}_{t=0}^{\infty}$  is a given nonstochastic nonnegative endowment sequence and  $\sum_{t=0}^{\infty} \beta^t y_t < \infty$ .
- Non-Ponzi-scheme condition

$$\lim_{T \rightarrow \infty} R^{-T} b_T = 0.$$

A particular assumption for this section:

$$\beta R = 1.$$

The steps to go through:

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<sup>3</sup>The notes in this section are adapted from LS 10.2 and Williamson 6.1.1..

1. Derive the intertemporal budget constraint using the forward operator and transversality condition:

$$\sum_{j=0}^{\infty} R^{-j} c_j = b_0 + \sum_{j=0}^{\infty} R^{-j} y_j.$$

2. Derive the value function and Euler equation.
3. Friedman's permanent income hypothesis and its implication.

### 3.1.1 Two types of Borrowing Constraints

Type 1: No-borrowing constraint

$$b_t \geq 0$$

Type 2: Natural borrowing constraint:

$$b_t \geq \bar{b}_t = - \sum_{j=0}^{\infty} R^{-j} y_{t+j}.$$

Even with  $c_t = 0$ , the consumer cannot repay more than  $\bar{b}_t$ .

### 3.1.2 Solutions to Consumption-Saving Decision

The first-order conditions for this problem are:

$$u'(c_t) \geq \beta R u'(c_{t+1}), \forall t \geq 0;$$

and

$$u'(c_t) > \beta R u'(c_{t+1}) \text{ implies } b_{t+1} = 0, \text{ not vice versa!}$$

Now consider the borrowing constraint  $b_{t+1} \geq 0$  for all  $t$ . Assume  $b_0 = 0$  for the following examples, and examine the optimal consumption path.

**Example 8** *The endowment path*  $\{y_t\}_{t=0}^{\infty} = \{y_h, y_l, y_h, y_l, \dots\}$

**Example 9** *The endowment path*  $\{y_t\}_{t=0}^{\infty} = \{y_l, y_h, y_l, y_h, \dots\}$

**Example 10** *Assume*  $y_t = \lambda^t$ , *and*  $1 < \lambda < R$ . *(the case of natural borrowers)*

**Example 11** Assume  $y_t = \lambda^t$ , and  $\lambda < 1$ .

**Remark 12** Given a borrowing constraint and a nonstochastic endowment stream, the impact of the borrowing constraint will not vanish until the household reaches the period with the highest annuity value of the remainder of the endowment stream.

**Remark 13** With a nonrandom endowment that does not grow perpetually, consumption does converge.

Now assume only the natural borrowing constraint.

**Example 14** Assume  $y_t = \lambda^t$ , and  $1 < \lambda < R$ .

**Remark 15** Perfect consumption smoothing is allowed over time under the natural borrowing constraint in general. (See LS 16.3 for a proof)

### 3.1.3 Empirical Evidence on Permanent Income Hypothesis<sup>4</sup>

Differences in the cross-sectional evidence and time-series evidence. Across households at a point in time, the relationship is indeed of the type that Keynes postulated, that is, consumption is determined by current disposable income. But within a country over time, aggregate consumption is essentially proportional to aggregate income.

The permanent-income hypothesis implies

$$Y = Y^p + Y^T,$$

where in most samples transitory income has a mean near zero and is roughly uncorrelated with permanent income.

Substituting the above equation into the following regression:

$$C_t = \beta_0 + \beta_1 y_t + \varepsilon_t,$$

and taking into account  $C = Y^p$ , we have

$$\beta_1 = \frac{\text{var}(Y^p)}{\text{var}(Y^p) + (Y^T)}.$$

Across households, much of the variation in income reflects factors as unemployment and the fact that households are at different points in their life cycles. As a result, the estimated slope coefficient is substantially less than 1. Over time, in contrast, almost all the variation in aggregate income reflects long-run growth, that is, permanent increases in the economy's resources.

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<sup>4</sup>Romer (2006) Chapter 7 has good discussion on this.

## 3.2 Consumption under Uncertainty<sup>5</sup>

Now we consider the case of consumption under uncertainty. First we start with some key definitions of stochastic processes.

### 3.2.1 Stochastic Processes

Martingale, stationarity, random walk

### 3.2.2 Hall (1978)'s Random Walk Model

A stochastic maximization problem as stated in (7) yields

$$E_t [U' (c_{t+1})] = U' (c_t). \quad (8)$$

The same maximization problem as in (7) except

- Utility function takes the linear-quadratic form:

$$u (c_t) = ac_t - bc_t^2.$$

We still maintain the assumption:

$$\beta R = 1.$$

The linearity of marginal utility in turn implies

$$E_t c_{t+1} = c_t.$$

The optimal path of consumption is such that consumption is expected to be constant over the remainder of his life span (Note: NOT constant over time!).

The property of *certainty equivalence*: The solution is the same as that which would obtain if there was no uncertainty, or if equivalently the individual held expectations of labor income with subjective certainty.

Implications:

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<sup>5</sup>This section is adapted from Romer (2006) Chapter 7, LS (2004) Chapter 16, and Blanchard and Fischer (1989) Chapter 6.

- Consumption smoothing (Note the difference between the realized budget constraint and the expected value budget constraint. the former implies the later but not the reverse. ):

$$c_0 = (1 - R^{-1}) \left[ b_0 + E_0 \left( \sum_{j=0}^{\infty} R^{-j} y_j \right) \right].$$

- Uncertainty about future income has no impact on consumption.
- Only unexpected changes in the future path of income have an impact on consumption:

$$c_t - c_{t-1} = (1 - R^{-1}) \left[ E_t \left( \sum_{j=0}^{\infty} R^{-j} y_{t+j} \right) - E_{t-1} \left( \sum_{j=0}^{\infty} R^{-j} y_{t+j} \right) \right]$$

It is easy to verify that if labor income follows a stationary first-order process with coefficient  $\rho$ , we have

$$c_t - c_{t-1} = \frac{1 - R^{-1}}{1 - \rho R^{-1}} \epsilon_t.$$

Thus the marginal propensity to consume out of income, measured as the change in consumption in response to an unexpected change in income, is given by  $\frac{1-R^{-1}}{1-\rho R^{-1}}$ , which is less than one. This is the result emphasized by both Friedman (1956) in the “permanent income hypothesis” and by Modigliani (1986) in the “life cycle” theory that consumption smooths transitory changes in income.

### 3.2.3 Empirical Test of Hall’s Model

Hall (1989) and Campbell and Mankiw’s (1989) test allowing the so-called rule-of-the thumb consumers.

## 3.3 Solutions to Excess Sensitivity Puzzle of Consumption

We will discuss several proposed solutions:

### 3.3.1 Precautionary Saving

Quadratic utility is an unattractive description of behavior toward risk. If utility is quadratic, marginal utility is linear ( $U''' = 0$ ) in consumption: an increasing the variance of consumption has no effect on expected marginal utility, and thus no effect on optimal behavior. But most plausible utility functions, that is utility functions that imply plausible behavior toward risk, are such that  $U''' > 0$ . This means that marginal utility is convex in consumption, and an increase in uncertainty raises the expected marginal utility.

To maintain equality in equation (8), the expected future consumption must increase compared to current consumption. Uncertainty leads consumers to defer consumption, to be more prudent.

Assume that the consumer maximizes

$$E_0 \left[ \sum_{t=0}^{T-1} \left( -\frac{1}{\alpha} \right) \exp(-\alpha C_t) \right]$$

subject to

$$A_{t+1} = A_t + Y_t - C_t,$$

and

$$Y_t = Y_{t-1} + e_t, e_t \sim N(0, \sigma^2).$$

The following assumptions hold:

- Constant absolute risk aversion.
- The subjective discount rate, is equal to the riskless interest rate, and they are both equal to zero.
- Labor income follows a random walk, with normally distributed innovations.

It is then easy to verify that optimal consumption satisfies

$$C_{t+1} = C_t + \frac{\alpha \sigma^2}{2} + e_{t+1},$$

and that the level of consumption is given by

$$C_t = \left( \frac{1}{T-t} \right) A_t + Y_t - \frac{\alpha(T-t-1)\sigma^2}{2}.$$

### 3.3.2 Liquidity Constraints

The presence of liquidity constraints causes individuals to save as insurance against the effects of future falls in income.

Assume that the consumer maximizes

$$E_0 \left[ \sum_{t=1}^3 (aC_t - bC_t^2) \right]$$

subject to

$$A_{t+1} = A_t + Y_t - C_t,$$

and  $Y_t$  is stochastic. The subjective discount rate, is equal to the riskless interest rate, and they are both equal to zero.

At period 3, we have

$$\begin{aligned} C_3 &= A_3 + Y_3 \\ &= A_2 + Y_2 + Y_3 - C_2. \end{aligned}$$

If the liquidity constraint does not bind, the consumer chooses

$$C_2 = \frac{A_2 + Y_2 + E_2 Y_3}{2}.$$

If the constraint binds, the consumer chooses

$$C_2 = A_2 + Y_2.$$

Thus,

$$C_2 = \min \left\{ \frac{A_2 + Y_2 + E_2 Y_3}{2}, A_2 + Y_2 \right\}.$$

Now consider the first period. If the liquidity constraint is not binding that period, we have our usual Euler equation

$$C_1 = E_1 C_2.$$

However, if the probability that the liquidity constraint will bind in the second period is strictly positive,  $C_1$  will be smaller compared to the case when such probability is zero. Thus even when the liquidity constraint does not bind currently, the possibility that it will bind in the future reduces consumption.

## 4 Asset Pricing

### 4.1 Lucas Asset Pricing Model

The economy consists of identical infinitely lived consumers. The representative consumer maximizes

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t U(C_t) \right].$$

The following assumptions hold:

- There are  $n$  risky assets in the economy, each of which generates a stochastic physical return in the form of perishable fruit, equal to  $d_{it}$  per period. (The assets can be thought of as trees, and output as seedless apples.)
- The assets are the only source of income in the economy.

Denote by  $p_{it}$  the ex-dividend price of asset  $i$  in period  $t$ . Let  $p_t$  and  $d_t$  be the  $n \times 1$  vectors of prices and dividends at time  $t$ . Let  $x_{it}$  be the quantity of asset  $i$  that the consumer holds between  $t$  and  $t + 1$ . Let  $x_t$  be the  $n \times 1$  vectors of  $x_{it}$ . The budget constraint is

$$C_t + p_t' x_t = (p_t + d_t)' x_{t-1}.$$

The Euler equations are

$$p_{it} = E_t \left[ \beta \frac{U'(C_{t+1})}{U'(C_t)} (p_{i,t+1} + d_{i,t+1}) \right]. \quad (9)$$

Assume that there is one unit of each asset, equilibrium implies that  $x_{it} = 1$  for all  $i, t$ . From the budget constraint this implies that

$$C_t = \sum d_{it}.$$

Assuming no bubbles<sup>6</sup>, equation (9) can be solved forward to get

$$p_{it} = E_t \left[ \sum_{j=1}^{\infty} \beta^j \frac{U'(C_{t+j})}{U'(C_t)} d_{i,t+j} \right]$$

The price is equal to the expected present discounted value of dividends, where the time-varying stochastic discount rate used for  $t+j$  is the marginal rate of substitution between consumption at time  $t+j$  and consumption at time  $t$ .

Now assume that there is only one tree, that is,  $n = 1$ .

**Example 16** *Assume that the consumer is risk neutral, that is,  $U(C) = C$ , prove that in this case movements in prices come from movements in expected dividends.*

**Example 17** *Assume that the consumer is risk averse with  $U(C) = \ln C$ , prove that the price-dividend ratio is a constant. Explain intuitively why price is a multiple of dividend in this case. (Explain how to obtain this solution using the Euler equation.)*

**Example 18** *Assume that the dividend process follows a Markov chain. In particular when output takes on two values (the example in Williamson's notes)*

**Example 19** *Assume that  $y_{t+1} = \lambda_{t+1} y_t$ , where  $\lambda_t$  is Markov with transition matrix  $P$ . Assume that  $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$ , explain how you would compute the equilibrium asset price.*

Now assume that the agents can borrow and lend. That is, there is a risk-free asset which trades on a competitive market at each date. This is a one-period risk-free bond which is a promise to pay one unit of consumption in the following period. Let  $b_t$  and  $q_t$  denote the quantity and price of risk-free

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<sup>6</sup>That is,

$$E_t \lim_{j \rightarrow \infty} \beta^j U'(C_{t+j}) p_{i,t+j} = 0.$$

If the bubble is positive, the marginal utility gain of selling shares,  $U'(C_t) p_{it}$ , exceeds the marginal utility loss of holding the asset forever and consuming the future stream of dividends. thus all agents would like to sell some of their shares and the price would be driven down.

bonds acquired in period  $t$  by the representative agent. The representative agent's budget constraint is then

$$C_t + p_t'x_t + q_t b_t = (p_t + d_t)'x_{t-1} + b_{t-1}.$$

In equilibrium, we will have  $b_t = 0$ , *i.e.*, there is a zero net supply of bonds, and prices need to be such that the bond market clears.

Compare the general equilibrium case with the partial equilibrium case when the interest rate is exogenously given and there exist an outside for risk-free borrowing and lending.

**Example 20** *What is  $q_t$ ? What is the one-period risk-free rate  $r_{f,t} = R_{f,t} - 1$ ? What is the risk-free interest rate if the representative agent is risk neutral?*

**Example 21** *Derive the equilibrium stock return assuming that utility is CRRA, and also that the rate of growth of consumption, defined as  $\log(C_{t+1}) - \log(C_t)$ , and the net rate of return on asset  $i$ , defined as  $\log(1 + R_{it})$ , are conditionally jointly normally distributed, with conditional mean  $g$  and  $x$  and covariance matrix  $V$ .*

**Example 22** *Using the fact that, for any two random variables,  $X$  and  $Y$ ,  $\text{cov}(X, Y) = E(XY) - E(X)E(Y)$ , the so-called covariance decomposition, to find the equity premium for each type of equity.*

The Euler equation can be written as

$$E_t(m_{t+1}R_{i,t+1}) = 1, \text{ where}$$

$$m_{t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)}, R_{i,t+1} = \frac{p_{i,t+1} + d_{i,t+1}}{p_{i,t}}.$$

Using the covariance decomposition, we have

$$E_t R_{i,t+1} - R_{f,t} = -R_{f,t} \text{cov}_t(m_{t+1}, R_{i,t+1}) \tag{10}$$

$$= -R_{f,t} \sigma_t(m_{t+1}) \sigma_t(R_{i,t+1}) \text{corr}_t(m_{t+1}, R_{i,t+1}) \tag{11}$$

The intertemporal marginal rate of substitution  $m_{t+1}$  is the stochastic discount factor. The covariance of the asset's return with the stochastic discount factor is called the asset's  $\beta$ -coefficient.

**Remark 23** *Shares with high expected returns are those for which the covariance of the asset's return with the intertemporal marginal rate of substitution is low. Alternatively, with diminishing marginal utility, the implication is that in equilibrium consumers are willing to accept a lower expected return on an asset that provides a hedge against low consumption by paying off more in states when consumption is low. That is, the representative consumer will pay a high price for an asset which is likely to have high payoffs when the marginal utility of consumption is high, or in other words, when aggregate consumption is low.*

**Remark 24** *A “risky” stock, one that has a high standard deviation,  $\sigma_t(R_{i,t+1})$ , may nonetheless command no greater average return than the risk-free rate if its return is uncorrelated with the stochastic discount factor.*

**Example 25** *Hint on the term structure of interest rates.*

## 4.2 The Issue of Equity Premium

Now we consider the unconditional version of the Euler equation. Assuming constant relative risk aversion, the equity premium is determined by

$$\gamma \text{cov}(\Delta C, R_i) = \gamma \sigma(\Delta C) \sigma(R_i) \text{corr}(\Delta C, R)$$

where  $\gamma$  is the relative risk aversion parameter. Higher  $\gamma$  means that more consumption gives less pleasure very quickly; it implies that people are less willing to substitute less consumption now for more consumption later and to take risks.

The issue: The standard deviation of aggregate consumption growth is about 1 percent. For a horizon of one year or so, the correlation of consumption growth with stock returns is 0.2 or so, and  $\sigma(R_i)$  is around 20 percent. Putting these together,  $\gamma$  must be uncharacteristically high to explain the equity premium. However, economists have shied away from high curvature  $\gamma$  on the basis that people do not seem that risk averse.

Mehra and Prescott example given in Williamson notes.

The roles played by  $\gamma$  in Mehra and Prescott's expected utility model:

1. The higher is  $\gamma$ , the lower is the intertemporal elasticity of substitution,  $\frac{1}{\gamma}$ , and the greater is the tendency of the representative consumer to smooth consumption over time. The average real risk free interest rate

has to be very high to rationalize the observed consumption growth rate in the data.

2. The higher is  $\gamma$ , the representative agent must be compensated more for bearing risk. The problem in terms of fitting the model is that there is not enough variability in aggregate consumption to produce a large enough risk premium, given plausible levels of risk aversion.

The issue of equity premium puzzle is more challenging in a production economy where consumption can be smoothed.

Proposals to deal with the equity premium puzzle: Non-expected utility, internal and external habit formation and etc.

## 5 Real Business Cycle Models

### 5.1 Mechanism and Calibration

The key idea of the RBC theory is that business cycle can arise in frictionless perfectly competitive and complete markets in which there are real or technology shocks. It is notable because its micro foundations are fully specified and it links the short-run with the neoclassical growth model. The typical RBC model has the structure of a standard neoclassical growth model with a labor/leisure choice incorporated. In a business cycle model, we need to specify the

- Impulse mechanism - the event that causes a variable to deviate from its steady state
- Amplification mechanism - the mechanism that causes deviation of output for given deviations of impulses.
- Propagation mechanism - the mechanism causing deviations from the steady state to persist.

The main propagation mechanisms are:

- Risk averse agents smooth consumption over time using capital.
- Lags in investment causes shocks to propagate.
- Agents substitute leisure in response to transitory changes in wages.

Calibration (Kyland and Prescott) : relying on microeconomic empirical studies and on the long-run properties of the economy to choose parameter values. Variance, co-variance and cross correlations for the simulated time series are compared to the comparable statistics in the data. The model is then judged to be close or not based on this comparison, so the approach differs significantly from standard econometrics.

## 5.2 Success and Drawbacks of RBC Models

### 5.2.1 Dynamics of the Benchmark RBC Model

The importance of impulse response functions. Computation of the impulse responses for AR, AR(p) and ARMA(p,q) process.

The issue with propagation in the RBC model: exogenous persistence instead of internal persistence. Remedy: habit formation, time to build and time to plan and etc.

The issue with amplification: the case with fixed labor versus flexible labor; the dynamic labor supply curve and the empirical measure of labor supply elasticity. Remedy: indivisible labor.

Comparison of the second moments observed in the data and generated by the model. The statistics is contained in table 1 and 3 from Chapter 14 of Handbook of Macroeconomics.

### 5.2.2 Success of the RBC models:

1. Productivity shocks produce a model economy that is nearly as volatile as the actual US economy. Kyland and Prescott (1991) argued that “technology shocks account for 70 percent of business cycle fluctuations.
2. Investment is about three times more volatile than output in both the actual economy and the model economy. However, consumption is only about one-third as volatile as output while it is over two thirds as volatile as output in the US economy.
3. The persistence generated by the basic model is high, but weaker than in the data.
4. The basic RBC model captures the general pattern of comovement in the data. However, the empirical correlations of output with labor, investment and productivity are substantially smaller than their model counterparts.

### 5.2.3 Major Criticism

The controversial approach of the RBC approach: Technology shocks are the dominant source of fluctuations. The major criticism:

1. The model's performance requires an empirically unreasonable degree of intertemporal substitution in labor supply.
2. The strongly procyclical character of the model's real wage rate is inconsistent with the finding of numerous micro empirical studies. The standard preferences are not compatible with the equity premium puzzle.
3. The use of the Solow residual was highly problematic, leading to excessively volatile productivity shocks.

The final criticism remains the Achilles heel of the RBC literature. Problems include labor hoarding and the cyclical variation in capital utilization. Solow-residual based measures of technology shocks that do not account for unmeasured variations in labor and capital will tend to be more volatile and procyclical than true shocks to technology. For this reason the residual was often referred to as a "measure of our ignorance". The relative standard deviations also provide a measure of the limited extent to which the basic RBC model amplifies productivity shocks: in term of its business cycle behavior, output is 1.48 times as volatile as productivity.

### 5.3 Variations of the RBC Model

Two shortcomings in the implications of real business cycle theory for the labor market:

1. Hours worked fluctuate considerably more than productivity.
2. The correlation between hours worked and productivity is close to zero.

The first fact implies that the short-run labor supply elasticity must be large.

The real business cycle model implies a large positive correlation between hours and productivity while the data display a near-zero correlation. This result arises because the model is driven by a single shock to the aggregate production function, which can be interpreted as shifting the labor demand curve along a stable labor supply curve and inducing a very tight positive relationship between hours and productivity.