Midterm Exam

Econ 306 (Spring 09)

Q1

(a) $$V(K_t, A_t, Q_t) = \max_{\{K_{t+1}, L_t\}} \left\{ u(C_t) + \beta E_t V(K_{t+1}, A_{t+1}, Q_{t+1}) \right\}$$

s.t. $$C_t + \frac{K_{t+1}}{Q_t} = A_t K_t^{\alpha} L_t^{1-\alpha}$$ ← Aggregate resource constraint

OR: $$V(K, A, Q) = \max_{\{K', L\}} \left\{ u(C) + \beta E\left[V(K', A', Q') \mid A, Q\right] \right\}$$

s.t. $$C + \frac{K'}{Q} = A K^{\alpha} L^{1-\alpha}$$

(b) FOC: $$\begin{cases} \dfrac{1}{C_t Q_t} = \beta E_t V_1(K_{t+1}, A_{t+1}, Q_{t+1}) \\ L_t = 1 \end{cases}$$

Env: $$V_1(K_t, A_t, Q_t) = \frac{1}{C_t} \cdot \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}$$

$\Rightarrow$ Euler: $$\frac{1}{C_t Q_t} = \beta E_t \left[\frac{\alpha A_{t+1} K_{t+1}^{\alpha-1}}{C_{t+1}} \right]$$

(c) $V(K,A,Q) = \max_{\{K',L\}} \left\{ u(C) + \beta E V(K', A', Q' | A, Q) \right\}$

Guess $V(K,A,Q) = G + H\ln K + F\ln A + D\ln Q$

FOC w/r. K': $\frac{1}{C_t Q_t} = \beta E_t \{ V_1(K_{t+1}, A_{t+1}, Q_{t+1}) \}$

$$\Rightarrow \frac{1}{C_t Q_t} = \frac{\beta H}{K_{t+1}}$$

According to the Aggregate resource constraint.

$$C_t Q_t + K_{t+1} = Q_t A_t K_t^\alpha$$

$$\Rightarrow \frac{1}{Q_t A_t K_t^\alpha - K_{t+1}} = \frac{\beta H}{K_{t+1}} \Rightarrow \begin{cases} K_{t+1} = \cancel{\alpha\beta Q_t A_t K_t^\alpha} \; \frac{\beta H}{1+\beta H} Q_t A_t K_t^\alpha \\ C_t = \cancel{(1-\alpha\beta) A_t K_t^\alpha} \; \frac{1}{1+\beta H} A_t K_t^\alpha \end{cases}$$

Substitute the solution to K_{t+1} into the rHs of Bellman Equation.

$$G + H\ln K_t + F\ln A_t + D\ln Q_t$$

$$= \log\left(\frac{1}{1+\beta H}\right) + \log A_t + \alpha \log K_t$$
$$+ \beta \Big[G + H\ln\left(\frac{\beta H}{1+\beta H}\right) + H\log Q_t + H\log A_t + \alpha H \log K_t$$
$$+ F E_t \ln A_{t+1} + D E_t \ln Q_{t+1} \Big]$$

Matching terms

$$H = \beta \alpha H + \alpha \Rightarrow H = \frac{\alpha}{1-\alpha\beta}$$

$$D = \beta H + \beta D \rho_Q \Rightarrow D = \frac{\alpha\beta}{(1-\alpha\beta)(1-\beta\rho_Q)}$$

$$F = 1 + \beta H + \beta F \rho_A \Rightarrow F = \frac{1}{(1-\alpha\beta)(1-\beta\rho_A)}$$

(d) $$\frac{C_t}{Y_t} = \frac{1}{1+\beta H} = 1 - \alpha\beta$$

$$\frac{K_{t+1}}{Y_t} = \frac{\beta H}{1+\beta H} = \alpha\beta Q_t$$

As $\alpha \uparrow$, MPK higher, more investment;

As $\beta \uparrow$, more weight on the future, more investment;

As $Q_t \uparrow$, the same amount of investment turns into larger amount of capital stock, more investment.

(e) Since ~~the~~ $\frac{C}{Y}$ is constant, the impulse responses of $\ln C$ and $\ln Y$ are the same.

For $\ln K_t$

$$\frac{\partial \ln K_t}{\partial \varepsilon_{A,t}} = 0$$

$$\frac{\partial \ln K_{t+1}}{\partial \varepsilon_{A,t}} = 1$$

$$\frac{\partial \ln K_{t+2}}{\partial \varepsilon_{A,t}} = \rho_A + \alpha \frac{\ln K_{t+1}}{\partial \varepsilon_{A,t}} = \rho_A + \alpha$$

$$\frac{\partial \ln K_{t+j}}{\partial \varepsilon_{A,t}} = \rho_A^{j-1} + \alpha \frac{\ln(K_{t+j-1})}{\partial \varepsilon_{A,t}}$$ Transition is on.

For $\ln C_t$, $\ln Y_t$,

$$\frac{\partial \ln C_t}{\partial \varepsilon_{A,t}} = 1$$

$$\frac{\partial \ln C_{t+j}}{\partial \varepsilon_{A,t}} = \rho_A^{j} + \frac{\alpha \ln K_{t+j}}{\partial \varepsilon_{A,t}}$$

~~For The impw~~ For $\varepsilon_{Q,t}$, $$\frac{\partial \ln K_{t+1}}{\partial \varepsilon_{Q,t}} = 1, \quad \frac{\partial \ln K_{t+j}}{\partial \varepsilon_{Q,t}} = \rho_Q^{j-1} + \alpha \frac{\partial \ln K_{t+j-1}}{\partial \varepsilon_{Q,t}}$$

$$\frac{\partial \ln C_t}{\partial \varepsilon_{Q,t}} = 0, \quad \frac{\partial \ln C_{t+j}}{\partial \varepsilon_{Q,t}} = \alpha \frac{\partial \ln K_{t+j}}{\partial \varepsilon_{Q,t}}$$

The impact die out over time

$u(C,l) = \ln C$ (5)

Q2.

(a) $$V(w) = \max_{\{accept,\ reject\}} \left\{ G(w),\ \overset{u(c,1)}{\cancel{0}} + \beta E V(w') \right\}$$

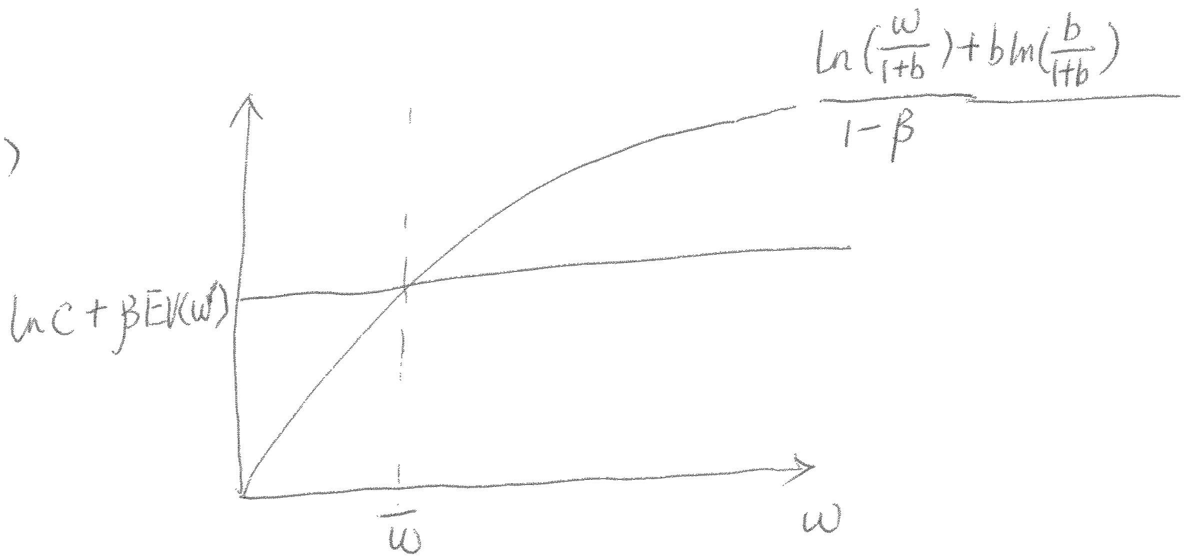
G(w): the maximum lifetime util if accepting the offer

$$G(w) = \max_{l} \left\{ u(y, l) + \beta G(w) \right\}$$

(b) where $y = w(1-l)$

FOC $\Rightarrow \dfrac{w}{y} = \dfrac{b}{l} \Rightarrow l = \dfrac{b}{1+b}$

(c)



(d)
$$\frac{\ln\left(\frac{\bar{w}}{1+b}\right) + b\ln\left(\frac{b}{1+b}\right)}{1-\beta} = \ln C + \beta E V(w')$$

Because
$$\begin{cases} V(w') = \dfrac{\ln\left(\frac{w'}{1+b}\right) + b\ln\left(\frac{b}{1+b}\right)}{1-\beta} & \text{if } w' > \bar{w} \\ \cancel{\ln C +} \ \dfrac{\ln\left(\frac{\bar{w}}{1+b}\right) + b\ln\left(\frac{b}{1+b}\right)}{1-\beta} & \text{if } w' < \bar{w} \end{cases}$$

$$\Rightarrow \frac{\ln\left(\frac{\bar{w}}{1+b}\right) + b\ln\left(\frac{b}{1+b}\right)}{1-\beta} = \ln C + \beta\left\{\int_0^{\bar{w}} \frac{\ln\left(\frac{\bar{w}}{1+b}\right) + b\ln\left(\frac{b}{1+b}\right)}{1-\beta} dF(w) + \int_{\bar{w}}^{B} \frac{\ln\left(\frac{w}{1+b}\right) + b\ln\left(\frac{b}{1+b}\right)}{1-\beta} dF(w)\right\}$$

Q3.

(a)
$$v(k) = \max_{k'} \left\{\theta^{*} \log c + \beta w(k')\right\}$$
$$w(k) = \max_{k'} \left\{\theta^{**} \log c + \beta V(k')\right\}$$

Guess $v(k) = F + N\log k$, $w(k) = G + H\log k$

$$c = k - k'$$

FOC w/r rhs of $v(k)$

$$\Rightarrow \frac{\theta^*}{c} = \frac{\beta H}{k'} \Rightarrow \frac{\theta^*}{k-k'} = \frac{\beta H}{k'}$$

$$\Rightarrow k'_v = \frac{\beta H k}{\theta^* + \beta H}$$

FOC w/r rhs of $w(k)$

$$\Rightarrow \frac{\theta^{**}}{c} = \frac{\beta N}{k'} \Rightarrow k'_w = \frac{\beta N k}{\theta^{**} + \beta N}$$

Substitute in the optimal k'

$$\begin{cases} F + N\log k = \theta^* \log\left[\left(1 - \frac{\beta H}{\theta^* + \beta H}\right)k\right] + \beta G + \beta H \log k \\ G + H\log k = \theta^{**} \log\left[\left(1 - \frac{\beta N}{\theta^{**} + \beta N}\right)k\right] + \beta F + \beta N \log k \end{cases}$$

$$\Rightarrow \begin{cases} N = \theta^* + \beta H \\ H = \theta^{**} + \beta N \end{cases} \Rightarrow \begin{cases} N = \frac{\theta^* + \beta\theta^{**}}{1-\beta^2} \\ H = \frac{\theta^{**} + \beta\theta^*}{1-\beta^2} \end{cases}$$

(b)

Since $\theta^{*} > \theta^{**}$

It is easy to verify that $N > H$

~~It~~ The Results indicate that the consumer eats more in even periods when the same amount of consumption gives him higher utility (i.e he is more hungry.)

Q4. Covered in class