1 Univariate Function

Definition

The derivative of a univariate function $f(x)$ evaluated at x_0 is denoted either by $f'(x)$ or $\frac{df(x)}{dx}$:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Rules of Differentiation

- Sum-Difference Rule
$$\frac{d(f(x) \pm g(x))}{dx} = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$$
- Scale Rule
$$\frac{d(k \cdot f(x))}{dx} = k \cdot \frac{df(x)}{dx}$$
- Product Rule
$$\frac{d(f(x) \cdot g(x))}{dx} = \frac{df(x)}{dx} \cdot g(x) + f(x) \cdot \frac{dg(x)}{dx}$$
- Quotient Rule
$$\frac{d(\frac{f(x)}{g(x)})}{dx} = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$
- Chain Rule
$$\frac{d(g(f(x)))}{dx} = g'(f(x)) \cdot f'(x)$$

Derivatives of some common functional forms

- Constant function $f(x) = k$
$$\frac{df(x)}{dx} = 0$$
- Power function $f(x) = x^\alpha$
$$\frac{df(x)}{dx} = \alpha \cdot x^{\alpha - 1}$$
- Exponential function $f(x) = e^{kx}$
$$\frac{df(x)}{dx} = k \cdot e^{kx}$$

- Natural log function $f(x) = \ln(x)$

$$\frac{df(x)}{dx} = \frac{1}{x}$$

Exercise 1: Find the derivative of each function

1. $f(x) = \frac{1}{x}$
2. $f(x) = \sqrt{x} + \ln ax$
3. $f(x) = \ln x - \ln(1-x)$
4. $f(x) = 4x^3 \ln x$
5. $f(x) = e^{3x^2-2x}$

The Second Derivative

The derivative of a function is itself a function. The derivative of the derivative of a function is that function's second derivative. Denoted as $f''(x)$ or $\frac{d^2f(x)}{dx^2}$

$$\frac{d^2f(x)}{dx^2} = \frac{d}{dx}(\frac{df(x)}{dx})$$

Example: Utility function $u(c) = \ln c$

$$u'(c) = \frac{1}{c}$$

$$u''(c) = -\frac{1}{c^2}$$

A function is **concave** over an interval if its second derivative is negative for all values in that interval. Example $f(x) = \ln x$, $f(x) = \sqrt{x}$.

A function is **convex** over an interval if its second derivative is positive for all values in that interval. Example $f(x) = \frac{1}{x}$, $f(x) = x^2$.

Exercise 2: Find the second derivative of each function

1. $f(x) = 9 - 3x + 7x^2 - x^3$
2. $f(x) = \ln 4x$
3. $f(x) = \frac{1}{x}$
4. $f(x) = \sqrt{x}$
5. $f(x) = x^2e^x$

2 Multivariate function

Definition

Multivariate function is a function with more than one arguments, for example function with two arguments $f(x, y)$.

The **partial derivative** of a multivariate function with respect to any one of its arguments is the rate of change of the value of that function due to a very small change in that argument *while all the other variables that are also arguments of this function are held constant. The partital derivative of function $f(x, y)$ with respect to x and y is denoted as $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$, respectively.*

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$
$$\frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}$$

When you take partial derivatives of one argument, you treat all the other arguments as constant. Here are some examples:

- $f(x, y) = 3x^2 + \ln y$

$$\frac{\partial f(x,y)}{\partial x} = 6x$$
$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{y}$$

- $f(x, y) = x^{0.3} y^{0.7}$

$$\frac{\partial f(x,y)}{\partial x} = 0.3 \cdot x^{-0.7} \cdot y^{0.7}$$
$$\frac{\partial f(x,y)}{\partial y} = x^{0.3} \cdot 0.7 \cdot y^{-0.3}$$

- $f(x, y) = \sqrt{x + 2y}$

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{2} \cdot \frac{1}{\sqrt{x + 2y}}$$
$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{2} \cdot \frac{1}{\sqrt{x + 2y}} \cdot 2$$

Second partial derivative and cross partial derivative

*A **second partial derivative** is a measure of how a parital derivative with respect to one argument of a multivariate function changes with a very small change in that argument.*

*A **cross partial derivative** is a measure of how a parital derivative taken with repect to one of the argument of the multivariate function varies with a very small change in another argument of that function.*

With the bivariate function $f(x, y)$ we have two second partial denoted by

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f(x,y)}{\partial x})$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f(x,y)}{\partial y})$$

and two cross paritial denoted by

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f(x,y)}{\partial y})$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f(x,y)}{\partial x})$$

Indeed these two cross partials are equal to each other.

The second partial and cross partial of the above examples

- $f(x, y) = 3x^2 + \ln y$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = 6$$
$$\frac{\partial^2 f(x,y)}{\partial y^2} = -\frac{1}{y^2}$$
$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x} = 0$$

- $f(x, y) = x^{0.3}y^{0.7}$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = 0.3 \cdot (-0.7) \cdot x^{-1.7} \cdot y^{0.7}$$
$$\frac{\partial^2 f(x,y)}{\partial y^2} = x^{0.3} \cdot 0.7 \cdot (-0.3) \cdot y^{-1.3}$$
$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x} = 0.3 \cdot x^{-0.7} \cdot 0.7 \cdot y^{-0.3}$$

Exercise 3: Find the partial derivatives of each function

1. $f(x, y) = x^4 - 6x^2y + 4y^3$
2. $f(x, y) = \ln 3x - \ln(5 - y)$
3. $f(x, y) = x^2y^2$
4. $f(x, y) = \sqrt{xy}$