Notes on Campbell's (1994) "Inspecting the Mechanism"

I. Log-linear Approximations

1) Cobb-Douglas Production Function

\[ Y_t = A_t^\alpha K_t^{1-\alpha} N_t^\alpha \]

Log-linear form with no need of approximation:

\[ Y_t = \alpha A_t + (1-\alpha) K_t + \alpha N_t \]

Note: \( n_t \equiv 0 \) when \( N_t = 1 \)

2) Taylor Approximation:

\[ Y_t = f(X_t) \]

Now rewrite \( f(X_t) = f(\exp(x_t)) \), \( x_t = \log X_t = g(x_t) \)

\[ f(X_t) = g(x_t) \approx g(\bar{x}) + g'(\bar{x})(x_t - \bar{x}) \]

Examples:
\[ Y_t = \exp(Y_t) \approx \exp(\bar{Y}) + \exp(\bar{Y})(Y_t - \bar{Y}) \]

Recognize that \( f(\bar{x}) = g(\bar{x}) = \bar{Y} = \exp(\bar{Y}) \),

\[ Y_t = f(X_t) \text{ can be approximated by} \]

\[ \bar{Y}(Y_t - \bar{Y}) = g'(\bar{x})(X_t - \bar{x}) \]

*Key issues: impulse responses; matching second moments.*

Dropping constants for convenience in this paper (to avoid it, change of variables)

\[ \Rightarrow \bar{Y}Y_t = g'(\bar{x})X_t \]

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An important approximation

when \( g \) is small, \( \log(1+g) \approx g \)

(after dropping constants) or: \( \exp(g)-1 \approx g \)

Examples:

1. \( C_t^{-\gamma} = \exp(-\gamma C_t) \approx -\bar{C} \gamma C_t \)

2. \( Y_t = C_t + I_t \Rightarrow Y_t \approx \bar{C} C_t + (1-\bar{C}) \bar{I}_t \)
Solve the stochastic growth model.

Definition: a steady-state or balanced growth path of the model, in which $A_t, K_t, Y_t$ and $C_t$ all grow at a constant common rate, $G$.

Step 0: Solve the Model

Euler equation:

$$ C_t = \beta E_t [C_{t+1} R_{t+1}] $$

where

$$ R_{t+1} = (1-\alpha) \left( \frac{A_{t+1}}{K_{t+1}} \right) ^ \alpha + (1-S) $$

Capital Transition equation:

$$ K_{t+1} = (1-S) K_t + Y_t - C_t $$

where

$$ Y_t = A_t ^ \alpha K_t ^ {1-\alpha} $$

Step 1: Derive the Steady State

From (5):

$$ G^* = \beta R $$

$$ \Rightarrow R = -\frac{1}{\alpha} \log \beta + \frac{1}{\alpha} \log Y $$

Risk-free rate puzzle

From (4):

$$ \frac{A_t}{K} = \left[ \frac{G^* / \beta - (1-S)}{1-\alpha} \right] ^ \alpha \approx \left[ \frac{-r+8}{1-\alpha} \right] ^ \frac{1}{\alpha} $$

Approximation 3: $R \approx 1 + \log(R) = 1 + r$
Equation (9) and (10) follow.

**Step 2. Log-linearization**

The capital accumulation equation:

$$K_{t+1} = (1 - S) K_t + Y_t - C_t$$

$$\Rightarrow \frac{K_{t+1}}{K_t} = (1 - S) + \frac{Y_t - C_t}{K_t} \cdot \frac{Y_t - C_t}{Y_t}$$

$$\Rightarrow \log \left[ \exp (\Delta K_{t+1}) - (1 - S) \right] = Y_t - K_t + \log \left[ 1 - \exp (C_t - Y_t) \right]$$

**LHS**

$$\approx \frac{\exp (\Delta k)}{\exp (\Delta k) - (1 - s)} \quad \Delta K_{t+1} = \frac{\exp (\Delta g)}{\exp (\Delta g) - (1 - s)} \quad \Delta K_{t+1}$$

$$\approx \frac{1 + g}{g + s} \Delta K_{t+1}$$

**RHS**

$$\approx Y_t - K_t + \frac{-\exp (\bar{C} - \bar{Y})}{1 - \exp (\bar{C} - \bar{Y})} (C_t - Y_t)$$

$$\approx \left[ 1 - \frac{r + s}{(1 - \alpha) (g + s)} \right] (C_t - Y_t)$$

Substitute in equation (11),

$$\Rightarrow K_{t+1} \approx \lambda_1 K_t + \lambda_2 A_t + (1 - \lambda_1 - \lambda_2) C_t \quad (13)$$
The Euler Equation

\[ C_t^{-\gamma} = \beta E_t \left[ C_{t+1}^{-\gamma} R_{t+1} \right] \]
\[ \Rightarrow -C_t^{-\gamma} C_t = \beta E_t \left\{ C_{t+1}^{-\gamma} R (-\gamma) C_{t+1} + C_{t+1}^{-\gamma} R_{t+1} \right\} \]

\[ \Rightarrow -\delta C_t = -\gamma E_t C_{t+1} + E_t R_{t+1} \]
\[ \Rightarrow E_t \Delta C_{t+1} = \frac{1}{\delta} E_t R_{t+1} \]

The Rate of Return Equation

\[ R_{t+1} = (1-\alpha) \left( \frac{A_{t+1}}{K_{t+1}} \right)^\alpha + (1-S) \]

\[ r_{t+1} = \log \left[ (1-\alpha) \exp \left\{ \alpha (A_{t+1} - K_{t+1}) \right\} + (1-S) \right] \]

\[ \alpha \left[ (1-\alpha) \left( \frac{A}{K} \right)^\alpha \right] \left( \frac{A}{K} \right)^\alpha \frac{1}{(1-\alpha) \left( \frac{A}{K} \right)^\alpha + (1-S)} \]

\[ = \frac{\alpha (r+S)}{1+r} (A_{t+1} - K_{t+1}) \]

Substitute in the above Euler equation.

\[ E_t \Delta C_{t+1} = \frac{1}{\delta} \cdot \frac{\alpha (r+S)}{1+r} E_t (A_{t+1} - K_{t+1}) \quad (17) \]
when \( N_t = 1 \), \( N_t = 0 \).

\[
y_t = \alpha a_t + (1 - \alpha) k_t
\]

To close the model:

\[
a_t = \phi a_{t-1} + \varepsilon_t, \quad -1 \leq \phi \leq 1 \tag{18}
\]

— to be continued