Notes on Campbell's (1994)

"Inspecting the Mechanism"

I. Log-linear Approximations

1 — Cobb-Douglas Production Function

$$Y_t = A_t^{\alpha} K_t^{1-\alpha} N_t^{\alpha}$$

Log-linear form with no need of approximation:

$$y_t = \alpha a_t + (1-\alpha) k_t + \alpha n_t$$

Note: $n_t \equiv 0$ when $N_t = 1$

2 — $$Y_t = f(X_t)$$

Taylor Approximation:

$$f(X_t) \approx f(\bar{X}) + f'(\bar{X})(X_t - \bar{X})$$

Now rewrite $f(X_t) = f[\exp(x_t)]$, $x_t = \log X_t$

$$= g(x_t)$$

$$f(X_t) = g(x_t) \approx g(\bar{x}) + g'(\bar{x})(x_t - \bar{x})$$

Examples:

$$Y_t = \exp(y_t) \approx \exp(\bar{y}) + \exp(\bar{y})(y_t - \bar{y})$$

Recognize that $f(\bar{X}) = g(\bar{x}) = \bar{Y} = \exp(\bar{y})$.

$Y_t = f(X_t)$ can be approximated by

$$\bar{Y}(y_t - \bar{y}) = g'(\bar{x})(x_t - \bar{x})$$

Key issues: impulse responses;
matching second moments.

1 Dropping constants for convenience in this paper
(To avoid it, change of variables)

$$\Rightarrow \quad \bar{Y} y_t = g'(\bar{x}) x_t$$

3 — An important approximation

when g is small, $\log(1+g) \approx g$

(after dropping constants) or: $\exp(g) - 1 \approx g$

Examples:

1. $C_t^{-\gamma} = \exp(-\gamma c_t) \approx -\bar{C}^{-\gamma} \gamma c_t$;
2. $Y_t = C_t + I_t \Rightarrow y_t \approx \frac{\bar{C}}{\bar{Y}} c_t + (1 - \frac{\bar{C}}{\bar{Y}}) i_t$

Solve the Stochastic Growth Model.

Definition: a steady-state or balanced growth path of the model, in which A_t, K_t, Y_t and C_t all grow at a constant common rate, G.

Step 0: Solve the Model

Euler equation: $C_t^{-\gamma} = \beta E_t[C_{t+1}^{-\gamma} R_{t+1}]$ (5) where

$$R_{t+1} \equiv (1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha} + (1-\delta) \quad (4)$$

Capital Transition equation: $K_{t+1} = (1-\delta)K_t + Y_t - C_t$ (2) where

$$Y_t = A_t^{\alpha} K_t^{1-\alpha} \quad (1)$$

Step 1: Derive the Steady State

From (5): $G^{\gamma} = \beta R$ (6)

$\Rightarrow r = -\log\beta + \gamma \log g$ (7) (risk free rate puzzle)

From (4): $\frac{A}{K} = \left[\frac{G^{\gamma}/\beta-(1-\delta)}{1-\alpha}\right]^{\frac{1}{\alpha}} \approx \left[\frac{r+\delta}{1-\alpha}\right]^{\frac{1}{\alpha}}$ (8)

Approximation ③: $R \approx 1 + \log(R) = 1 + r$

Equation (9) and (10) follow.

Step 2. Log-linearization

$\boxed{\text{The capital accumulation equation:}}$

$$K_{t+1} = (1-\delta) K_t + Y_t - C_t$$

$$\Rightarrow \frac{K_{t+1}}{K_t} = (1-\delta) + \frac{Y_t}{K_t} \cdot \frac{(Y_t - C_t)}{Y_t}$$

$$\Rightarrow \log\left[\exp(\Delta k_{t+1}) - (1-\delta)\right] = y_t - k_t + \log\left[1 - \exp(c_t - y_t)\right]$$

$$LHS \approx \frac{\exp(\overline{\Delta k})}{\exp(\overline{\Delta k}) - (1-\delta)} \Delta k_{t+1} = \frac{\exp(g)}{\exp(g) - (1-\delta)} \Delta k_{t+1}$$

$$\approx \frac{1+g}{g+\delta} \Delta k_{t+1}$$

$$RHS \approx y_t - k_t + \frac{-\exp(\bar{c} - \bar{y})}{1 - \exp(\bar{c} - \bar{y})} (c_t - y_t)$$

$$\approx \left[1 - \frac{r+\delta}{(1-\alpha)(g+\delta)}\right] (c_t - y_t)$$

Substitute in equation (11),

$$\Rightarrow k_{t+1} \approx \lambda_1 k_t + \lambda_2 a_t + (1 - \lambda_1 - \lambda_2) c_t \qquad (13)$$

$\boxed{\text{The Euler Equation}}$

$$C_t^{-\gamma} = \beta E_t \left[C_{t+1}^{-\gamma} R_{t+1} \right]$$

$$\approx \qquad\qquad \approx$$

$$-\bar{C}^{-\gamma} \gamma c_t = \beta E_t \left\{ \bar{C}'^{-\gamma} \bar{R} (-\gamma) c_{t+1} + \bar{C}'^{-\gamma} \bar{R} r_{t+1} \right\}$$

$$\Rightarrow \quad -\gamma c_t = -\gamma E_t c_{t+1} + E_t r_{t+1}$$

$$\Rightarrow \quad E_t \Delta c_{t+1} = \frac{1}{\gamma} E_t r_{t+1}$$

$\boxed{\text{The Rate of Return Equation}}$

$$R_{t+1} \equiv (1-\alpha) \left(\frac{A_{t+1}}{K_{t+1}} \right)^{\alpha} + (1-\delta)$$

$$r_{t+1} \equiv \log\left[(1-\alpha) \exp[\alpha (a_{t+1} - k_{t+1})] + (1-\delta) \right]$$

$$\propto \frac{(1-\alpha)\left(\frac{A}{K}\right)^{\alpha} \alpha}{(1-\alpha)\left(\frac{A}{K}\right)^{\alpha} + (1-\delta)} (a_{t+1} - k_{t+1})$$

$$= \frac{\alpha (r+\delta)}{1+r} (a_{t+1} - k_{t+1})$$

Substitute in the above Euler equation.

$$E_t \Delta c_{t+1} = \frac{1}{\gamma} \frac{\alpha (r+\delta)}{(1+r)} E_t (a_{t+1} - k_{t+1}) \qquad (17)$$

when $N_t \equiv 1$, $n_t \equiv 0$.

$$y_t = \alpha a_t + (1-\alpha) k_t$$

To close the model:

$$a_t = \phi a_{t-1} + \varepsilon_t, \quad -1 \le \phi \le 1 \qquad (18)$$

— to be continued