TRANSITION LAYER SOLUTIONS OF A HIGHER ORDER EQUATION IN AN INFINITE TUBE

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1 Introduction

We are concerned with the higher order variational problem

$$I = \inf_{u \in \mathcal{A}} \int_{\Omega} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^{\alpha} u D^{\beta} u + W(u) \right] dx. \quad (1.1)$$

The integral in the right side of (1.1) is denoted by $E(u)$, the energy of u , in this paper. This problem has its origin in phase transition theory. The function u is the phase field of a binary material, and $E(u)$ represents the total energy of the material, assuming the temperature is constant. This energy has two parts: the interaction energy and the bulk energy.

$$\int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^{\alpha} u D^{\beta} u du \quad (1.2)$$

represents the interaction energy of the phase field and

$$\int_{\Omega} W(u) dx \quad (1.3)$$

represents the bulk energy of the phase field. Here we use multi-index notation: $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $D^{\alpha} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_d}^{\alpha_d}$. We assume the matrix $[A_{\alpha\beta}]$ is symmetric and positive definite, which, as we show below, arises naturally in the derivation of (1.1).

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The bulk energy density function W is assumed to be a balanced double well potential function, e.g. $W(u) = (u^2 - 1)^2$. W has two global minima scaled to be at -1 and 1 . We assume W is continuous on R and C^2 near -1 and 1 , $W \geq 0$, -1 and 1 are the only global minima of W , and $W(-1) = W(1) = 0$, $W''(-1) > 0$ and $W''(1) > 0$. Concerning the behavior of W near infinity, we assume that

$$\liminf_{y \rightarrow -\infty} W(y) > 0 \text{ and } \liminf_{y \rightarrow \infty} W(y) > 0.$$

We also need a key assumption

$$N > \frac{d+2}{4} \tag{1.4}$$

on the order of the problem.

In this paper we study the variational problem in a d -dimensional tube of infinite length, so we will take the physical domain Ω to be an infinite tube. Because we are interested in the transitions of the phase field, we assume that $\mathcal{A}$ is a class of functions that take values -1 at one end of the tube and 1 at the other end. The precise definitions of Ω and $\mathcal{A}$ will be given later. Let us first recall how (1.1) is derived from a microscopic viewpoint.

In the theory of phase transitions for binary materials from a microscopic viewpoint, one considers a state to be a distribution u over a lattice so that $u(r)$ is the probability that a solute atom occurs at site r . The Helmholtz free energy of a state is given by

$$F = H - TS$$

where $H(u)$ is the internal interaction energy, T is the absolute temperature and $S(u)$ is the entropy. Following Khachaturyan (see [8] and [9] for instance), the internal energy takes the form

$$H(u) = -\frac{1}{2} \sum_{r,r'} J(r-r') u(r) u(r')$$

where the sum is over the lattice sites and $J(r-r') = J(r'-r) > 0$ gives the pairwise interaction energies.

The total entropy takes the form of a sum over the lattice of

$$-K[u(r) \log u(r) + (1-u(r)) \log(1-u(r))],$$

where K is a positive constant, which has minima at 0 and 1 and which naturally penalizes values of u outside the interval $[0, 1]$.

In the analysis to follow we will consider a rearrangement of (1.1) changing H to

$$\frac{1}{4} \sum_{r,r'} J(r-r') (u(r) - u(r'))^2$$

so that F is written

$$\begin{aligned} & \frac{1}{4} \sum_{r,r'} J(r-r') (u(r) - u(r'))^2 \\ & + \sum_r (TK[u(r) \log u(r) + (1-u(r)) \log(1-u(r))] - cu^2(r)), \end{aligned}$$

where $c = (1/4) \sum_r J(r)$, which we assume is finite. Note that for TK small the second term is the sum over the lattice of a double well function of u . We will write this term as

$$\frac{1}{4} \sum_r W(u(r))$$

and assume that W is everywhere defined. For computational ease later, we re-scale u so that the minima of the double well function occur at $u = -1, 1$. Taking the continuum limit gives

$$F(u) = \frac{1}{4} \left[\int \int J(x-y) (u(x) - u(y))^2 dx dy + \int W(u(x)) dx \right].$$

We assume J is non-negative, symmetric about 0, and $\int J(x) dx = 1$.

As was done in [2], we change variables in the first integral using $\eta = (x - y)/2$, $\xi = (x + y)/2$, then expand $u(x) = u(\xi + \eta)$ and $u(y) = u(\xi - \eta)$ about ξ to get

$$2^d \int \int J(2\eta) \left(\sum_{k=1}^{\infty} \sum_{|\alpha|=2k-1} \frac{1}{\alpha!} D^\alpha u(\xi) \eta^\alpha \right)^2 d\xi d\eta,$$

where both integrals are over R^d , $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$, $D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_d}^{\alpha_d}$, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_d!$, and $\eta^\alpha = \eta_1^{\alpha_1} \eta_2^{\alpha_2} \dots \eta_d^{\alpha_d}$.

We truncate the infinite sum at $k = N$ and integrate with respect to η . This “truncated” free energy is given by (1.1) where

$$A_{\alpha\beta} = 2^d \int \frac{J(2\eta) \eta^{\alpha+\beta}}{\alpha! \beta!} d\eta.$$

Note that $[A_{\alpha\beta}]$ is positive definite since

$$\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} \xi_\alpha \xi_\beta = 2^d \int J(2\eta) \left(\sum_{i=1}^N \sum_{|\alpha|=2i-1} \frac{\eta^\alpha \xi_\alpha}{\alpha!} \right)^2 d\eta \geq 0,$$

and the equality holds only if $\xi_\alpha = 0$ for all α .

The Euler-Lagrange equation of (1.1) is

$$\begin{cases} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} 2A_{\alpha\beta} D^{\alpha+\beta} u - W'(u) = 0 \text{ in } \Omega, \\ B_j(u) = 0 \text{ on } \partial\Omega, \quad j = 1, 2, \dots, 2N-1; \end{cases} \quad (1.5)$$

where the B_j 's are the natural boundary conditions. The expressions of the B_j 's are complicated. In the case $d = 2$, $N = 2$ and the material is isotropic, i.e., $J(x) = J(|x|)$, we have

$$B_1(u) = \frac{b}{16} \frac{\partial u}{\partial \nu} + \frac{c}{3} \nu_1^2 \frac{\partial u_{yy}}{\partial \nu} + \frac{c}{3} \nu_2^2 \frac{\partial u_{xx}}{\partial \nu}$$

$$\begin{aligned}
& +\nu_1^2\left(\frac{5c}{9}\nu_1 u_{xxx} + c\nu_2 u_{xxy}\right) + \nu_2^2\left(\frac{5c}{9}\nu_2 u_{yyy} + c\nu_1 u_{xyy}\right) \\
B_2(u) &= \frac{\partial}{\partial \tau} \left[-\frac{10c}{9}\nu_1^2\nu_2 u_{xxx} - 2c\nu_1\nu_2^2 u_{xxy} + \frac{10c}{9}\nu_1^2\nu_2 u_{yyy} \right. \\
& \quad \left. + 2c\nu_1^2\nu_2 u_{xyy} + \frac{2c}{3}\nu_1\nu_2 \frac{\partial u_{xx}}{\partial \nu} - \frac{2c}{3}\nu_1\nu_2 \frac{\partial u_{yy}}{\partial \nu} \right] \\
& - \nu_1^2\left(\frac{5c}{9}u_{xxx} + cu_{xxy}\right) - \nu_2^2\left(\frac{5c}{9}u_{yyy} + cu_{xyy}\right) - \frac{c}{3}\nu_2^2\Delta u_{xx} - \frac{c}{3}\nu_1^2\Delta u_{yy}\nu_1^2 - \frac{b}{16}\Delta u \\
B_3(u) &= \frac{\partial^2}{\partial \tau^2} \left[\nu_2^2\left(\frac{5c}{9}\nu_1 u_{xxx} + c\nu_2 u_{xxy}\nu_2\right) + \nu_1^2\left(\frac{5c}{9}\nu_2 u_{yyy} + c\nu_1 u_{xyy}\right) \right. \\
& \quad \left. + \nu_1^2\left(\frac{c}{3}\frac{\partial u_{xx}}{\partial \nu}\right) + \nu_2^2\left(\frac{c}{3}\frac{\partial u_{yy}}{\partial \nu}\right) + \frac{b}{16}\frac{\partial u}{\partial \nu} \right] \\
& + \frac{\partial}{\partial \tau} \left[\nu_1\nu_2\left(\frac{5c}{9}u_{xxx} + cu_{xxy}\right) - \nu_1\nu_2\left(\frac{5c}{9}u_{yyy} + cu_{xyy}\right) + \frac{c}{3}\nu_1\nu_2(-\Delta u_{xx} + \Delta u_{yy}) \right] \\
& + \nu_1\left(\frac{5c}{9}u_{xxxx} + cu_{xxxxy}\right) + \nu_2\left(\frac{5c}{9}u_{yyyy} + cu_{xyyy}\right) \\
& + \frac{c}{3}\Delta u_{xxy}\nu_2 + \frac{c}{3}\Delta u_{xyy}\nu_1 + \frac{b}{8}\frac{\partial(\Delta u)}{\partial \nu} + \frac{a}{2}\frac{\partial u}{\partial \nu}
\end{aligned}$$

where a, b, c are three constants, $\nu = (\nu_1, \nu_2)$ is the normal vector to $\partial\Omega$ pointing outward, and $\tau = (\nu_2, -\nu_1)$ is the unit tangential vector that makes (τ, ν) positively oriented.

This paper is devoted to the existence of the energy minimizers of (1.1), or the solutions of (1.5). The simplest case $\Omega = R$ and $N = 1$ gives the equation

$$\begin{cases} u'' - W'(u) = 0 & \text{in } R \\ \lim_{x \rightarrow -\infty} u(x) = -1, \quad \lim_{x \rightarrow \infty} u(x) = 1. \end{cases} \quad (1.6)$$

It was shown by Fife and McLeod in [6] that this equation has a unique solution up to translation and the manifold of all the translates is globally stable. Indeed they considered the more general case where W is not necessarily balanced and they considered traveling wave solutions. Note that if W is balanced, the traveling wave solutions become standing waves (the speed is 0).

Gardner and Jones studied the case where $\Omega = R$ and $N = 2$ in [7]. They also considered traveling wave solutions when W is not necessarily balanced. Bates, Fife, Gardner and Jones [3] generalized this case to arbitrary N . In both papers the higher order equations were treated as the perturbations of the second order equation and the existence and stability of traveling wave solutions was obtained if the coefficients of the higher order derivatives were “small”.

In [5], the authors studied the case where W is balanced, $\Omega = R$, and N is arbitrary, but without the smallness condition on the higher order derivatives. We proved the existence of energy minimizing solutions having limits of $-1, 1$ at $-\infty, \infty$, by using the concentration-compactness principle to capture an energy minimizer from any minimizing sequence.

In this paper we consider a domain Ω which is a tube of infinite length in R^d . We will consider two kinds of tubes. In section 3, we study tubes which are periodic in one direction. The discrete translational symmetry allows us to prove the existence of the minimizers by taking appropriate translates of terms in a minimizing sequence. Section 4 contains the proof of a lemma used in section 3. Then in section 5, we consider tubes whose tails are “half-periodic”, and we give a sufficient condition on the geometry of the tube for the existence of energy minimizers. We also give various examples where such a condition is satisfied.

2 Concentration functions of $\{W(u_n)\}$

Ω is assumed to be a sufficiently smooth open connected set in R^d . We will sometimes treat Ω as a metric space equipped with a distance function $\rho : \Omega \times \Omega \rightarrow R$ defined by

$$\rho(x, y) = \inf_{\gamma \in \Gamma} \int_0^1 |\gamma'(t)| dt \quad (2.1)$$

where Γ is the class of all smooth paths $\gamma : [0, 1] \rightarrow \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$. We define

$$B_\Omega(x, r) = \{y \in \Omega : \rho(y, x) < r\} \quad (2.2)$$

to be the ball centered at x with radius r in the metric space (Ω, ρ) . If we denote the usual ball in R^d centered at x with radius r by $B(x, r)$, in general $B_\Omega(x, r) \neq B(x, r) \cap \Omega$.

To say Ω is a tube, we assume that: (1) there is a large ball B in R^d such that $\Omega \setminus B$ consists of two unbounded connected open sets Ω_1 and Ω_2 ; (2) there exist P_i and Q_i such that $P_i \subset \Omega_i \subset Q_i$ where P_i and Q_i are diffeomorphic to the half infinite cylinder $[0, \infty) \times D \subset R^d$, with D being the unit ball in R^{d-1} ; (3) the diffeomorphisms have bounded gradients. This means that neither end of the tube will shrink to a point at infinity. We assume throughout this paper that Ω is a tube.

To prescribe the condition of the phase fields at the two ends of the tube Ω , we denote the two ends of Ω by E_1 and E_2 . We use $x \rightarrow E_i$ to denote x approaching E_i uniformly.

To define the function class $\mathcal{A}$ in (1.1), we first state the following lemma.

Lemma 2.1 *If u is a function in $L^1_{loc}(\Omega)$ with $W(u) \in L^1(\Omega)$, $D^\alpha u \in L^2$ for all α , $|\alpha| = 2i - 1$, $i = 1, 2, \dots, N$, then $\lim_{x \rightarrow E_1} u(x)$ and $\lim_{x \rightarrow E_2} u(x)$ exist and equal -1 or 1 .*

Proof. Recall an interpolation inequality for intermediate derivatives (see theorem 4.14 [1]): Let $\Omega \subset R^d$ have the uniform cone property, and let ϵ_0 be a finite

positive number. Then there exists a constant $K = K(\epsilon_0, m, p, \Omega)$ such that for any ϵ , $0 < \epsilon \leq \epsilon_0$, any integer j , $0 \leq j \leq m-1$, and any $v \in W^{m,p}(\Omega)$

$$|v|_{j,p} \leq K\epsilon|v|_{m,p} + K\epsilon^{-j}(m-j)|v|_{0,p} \quad (2.3)$$

where

$$|v|_{j,p} = \left\{ \sum_{|\alpha|=j} \int_{\Omega} |D^{\alpha}v|^p dx \right\}^{1/p}.$$

With the help of this inequality, a standard smoothing argument implies that for every u in lemma 2.1, $D^{\alpha}u \in L^2(\Omega)$, $|\alpha| = 1, 2, \dots, 2N-1$. We now apply the Sobolev embedding theorem (see theorem 5.4 [1]) to conclude that there is σ , $0 < \sigma < 1$, such that

$$\sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^{\sigma}} < \infty. \quad (2.4)$$

Here we used our key assumption (1.4). Therefore we have a uniform bound on the modulus of continuity of u . Since $\int_{\Omega} W(u) < \infty$, we deduce that $\lim_{x \rightarrow E_1} u(x)$, $\lim_{x \rightarrow E_2} u(x)$ exist and equal -1 or 1 . $\square$

We now define

$$\mathcal{A}^* = \{u \in L^1_{loc}(\Omega) : D^{\alpha}u \in L^2(\Omega), |\alpha| = 2i-1, i = 1, \dots, N, \int_{\Omega} W(u) < \infty\}.$$

In view of lemma 2.1, we can decompose $\mathcal{A}^*$ in to four mutually disjoint classes, $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ and $\mathcal{A}'''$, where

$$\begin{cases} \mathcal{A} = \{u \in \mathcal{A} : \lim_{x \rightarrow E_1} u(x) = -1 \quad \lim_{x \rightarrow E_2} u(x) = 1\}, \\ \mathcal{A}' = \{u \in \mathcal{A} : \lim_{x \rightarrow E_1} u(x) = 1 \quad \lim_{x \rightarrow E_2} u(x) = -1\}, \\ \mathcal{A}'' = \{u \in \mathcal{A} : \lim_{x \rightarrow E_1} u(x) = 1 \quad \lim_{x \rightarrow E_2} u(x) = 1\}, \\ \mathcal{A}''' = \{u \in \mathcal{A} : \lim_{x \rightarrow E_1} u(x) = -1 \quad \lim_{x \rightarrow E_2} u(x) = -1\}. \end{cases} \quad (2.5)$$

We are only interested in the first class, $\mathcal{A}$.

We now give a necessary condition for a phase transition to occur.

Lemma 2.2 *For a given energy minimizing sequence $\{u_n\}$ and every $\epsilon_1 > 0$, $L > 0$ there exists $\delta > 0$ so that for every n and $B_{\Omega}(y, r) \subset \Omega$, with $L < r \leq \infty$,*

$$\int_{B_{\Omega}(y, r)} W(u_n) dx < \delta$$

implies that either

$$u_n(x) \in (-1 - \epsilon_1, -1 + \epsilon_1)$$

for all $x \in B_{\Omega}(y, r)$ or

$$u_n(x) \in (1 - \epsilon_1, 1 + \epsilon_1)$$

for all $x \in B_{\Omega}(y, r)$.

Proof. Suppose the contrary. There exist $\epsilon_1, L, \delta_k, \delta_k \rightarrow \infty$ as $k \rightarrow \infty$, u_{n_k} from the minimizing sequence, $B_\Omega(y_k, r_k)$, $L \leq r_k \leq \infty$, and $x_k, x_k \in B_\Omega(y_k, r_k)$, such that

$$\int_{B_\Omega(y_k, r_k)} W(u_{n_k}) dx \leq \delta_k \rightarrow 0 \text{ as } k \rightarrow \infty \quad (2.6)$$

and $u_{n_k}(x_k) \notin (-1 - \epsilon_1, -1 + \epsilon_1) \cup (1 - \epsilon_1, 1 + \epsilon_1)$. Because we have a bound independent of k on the moduli of continuity of the u_{n_k} 's, (2.4), we can find η independent of k such that for all $x \in B_\Omega(x_k, \eta)$, $|u_{n_k}(x) - 1| \geq \epsilon_1/2$ and $|u_{n_k}(x) + 1| \geq \epsilon_1/2$, so we deduce

$$\begin{aligned} \int_{B_\Omega(y_k, r_k)} W(u_{n_k}) dx &\geq \int_{B_\Omega(x_k, \eta) \cap B_\Omega(y_k, r_k)} W(u_{n_k}) dx \\ &\geq C\eta^d \min_{x \in B_\Omega(x_k, \eta) \cap B_\Omega(y_k, r_k)} W(u_{n_k}(x)) \\ &\geq C\eta^d \min_{y \in R \setminus (-1 - \epsilon_1/2, -1 + \epsilon_1/2) \cup (1 - \epsilon_1/2, 1 + \epsilon_1/2)} W(y) \geq C' > 0 \end{aligned}$$

where C depends on d only. This contradicts (2.6). $\square$

Our next lemma asserts that $\{\int_\Omega W(u_n)\}$ does not vanish.

Lemma 2.3 *For any energy minimizing sequence $\{u_n\} \subset \mathcal{A}$ there exists $\Lambda \in (0, \infty)$ such that for all n*

$$\int_\Omega W(u_n) dx \geq \Lambda.$$

Proof. Since $I < \infty$, $\limsup_{n \rightarrow \infty} \int_\Omega W(u_n) dx < \infty$. For every $u \in \mathcal{A}$ there is $x_n \in \Omega$ such that $u_n(x_n) = 0$. (2.4) yields a bound independent of n on the moduli of continuity of the u_n 's. Hence there exist $\epsilon_1 > 0$, $\delta > 0$, both independent of n , such that $u_n(x) \in (-\epsilon_1, \epsilon_1)$ for $x \in B_\Omega(x_n, \delta)$. This implies $\Lambda > 0$. $\square$

We now apply the concentration-compactness principle to $\{W(u_n)\}$. In lemma 2.3, we may take

$$\Lambda = \liminf_{n \rightarrow \infty} \int_\Omega W(u_n) dx \quad (2.7)$$

and then take $\{u_n\}$ to be a subsequence along which this is achieved. We treat $\{W(u_n)\}$ as a sequence of non-negative functions in $L^1(\Omega)$. Define the concentration functions

$$Q_n(t) = \sup_{y \in \Omega} \int_{B_\Omega(y, t)} W(u_n(x)) dx \quad (2.8)$$

for $t \geq 0$. $\{Q_n\}$ is a sequence of non-negative, non-decreasing, uniformly bounded functions on $[0, \infty)$ and

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} Q_n(t) = \Lambda. \quad (2.9)$$

By a classical result about monotone functions, there exist a subsequence of u_n , again denoted by u_n , and a non-negative, non-decreasing function $Q(t)$ such that

$$\lim_{n \rightarrow \infty} Q_n(t) = Q(t)$$

for all $t \geq 0$. Let

$$\lambda = \lim_{t \rightarrow \infty} Q(t). \quad (2.10)$$

Then $\lambda \in [0, \Lambda]$. Furthermore by (2.9) and (2.10) we may make the following three statements:

I: *If $\lambda = \Lambda$, there exist a subsequence of u_n , again denoted by u_n , and $y_n \in \Omega$ such that for every $\epsilon > 0$ there exist $r > 0$ and a positive integer n_1 so that*

$$\int_{B_\Omega(y_n, r)} W(u_n(x)) dx \geq \Lambda - \epsilon$$

for $n \geq n_1$.

II: *If $\lambda = 0$, then for all $r > 0$,*

$$\lim_{n \rightarrow \infty} \sup_{y \in \Omega} \int_{B_\Omega(y, r)} W(u_n(x)) dx = 0.$$

III: *If $\lambda \in (0, \Lambda)$, then there exists a subsequence of $\{u_n\}$ again denoted by $\{u_n\}$ such that for every $\epsilon > 0$ there exist arbitrarily large $r > 0$, a sequence $r_n \rightarrow \infty$ as $n \rightarrow \infty$, and a positive integer n_1 such that for $n \geq n_1$*

$$\lambda - \epsilon < Q_n(r), \quad Q_n(r_n) < \lambda + \epsilon.$$

These three statements can be found in the proof of lemma I.1 [10].

3 An existence theorem for periodic Ω

Throughout section 3, we assume the tube Ω is periodic in one direction. This translation invariance enables us to construct an energy minimizer.

We suppose there is $T \in \mathbb{R}^d$ such that $\Omega + T = \Omega$ where $\Omega + T = \{x + T : x \in \Omega\}$ is the translation of Ω by T . To say Ω is a tube, we assume the cross sections of Ω perpendicular to T are bounded (see figure 1). Our main result in this section is the following theorem.

Theorem 3.1 *If the tube Ω is periodic, every energy minimizing sequence of (1.1) gives rise to an energy minimizer in $\mathcal{A}$. Indeed if $\{u_n\}$ is an energy minimizing sequence, there exist a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, and a sequence of integers $\{i_n\}$ such that $\{u_n(\cdot + i_n T)\}$ converges in a suitable sense to an energy minimizer.*

Figure 1: An example of periodic Ω .

We will prove theorem 3.1 by showing that only the case $\lambda = \Lambda$ in statement I in section 2 can occur when $\{u_n\}$ is an energy minimizing sequence.

Lemma 3.2 *The case $\lambda = 0$ can not occur.*

Proof. As in the proof of lemma 2.3, we can find x_n , and δ independent of n , such that $\int_{B_\Omega(x_n, \delta)} W(u_n) dx \geq C$ for some C independent of n . $\square$

Lemma 3.3 *The case $\lambda \in (0, \Lambda)$ can not occur.*

The proof of this lemma is postponed to section 4.

Lemma 3.4 *The case $\lambda = \Lambda$ implies that there is $u \in \mathcal{A}$ such that*

$$E(u) = \inf_{v \in \mathcal{A}} E(v).$$

Proof. Statement I in section 2 implies that for every $\epsilon > 0$ there exist $\{y_n\}$, $r > 0$ and a positive integer n_1 such that for $n \geq n_1$

$$\int_{B_\Omega(y_n, r)} W(u_n) \geq \Lambda - \epsilon.$$

Using the periodicity of Ω and replacing r by $r + D_T$, we can translate the u_n 's by $i_n T$ where the i_n 's are some integers, T is the period of the tube and D_T is the diameter of a piece of the tube of "length" T , and therefore take y_n to be a point y independent of n . By further enlarging r , we may assume that Ω is divided into mutually disjoint open sets Ω_1 , Ω_2 and $B_\Omega(y, r)$ with

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{B_\Omega(y, r)}.$$

Assume E_1 is associated to Ω_1 and E_2 is associated to Ω_2 . For large n

$$\int_{\Omega_1} W(u_n(x + i_n T)) dx < 2\epsilon, \quad \int_{\Omega_2} W(u'_n(x + i_n T)) dx < 2\epsilon.$$

Lemma 2.2 then implies that the $u_n(\cdot + i_n T)$'s stay close to -1 in Ω_1 and close to 1 in Ω_2 uniformly in n . We pass to the limit along a subsequence and obtain $u \in \mathcal{A}$ such that

$$u_n(\cdot + i_n T) \rightarrow u \text{ in } C(K \cap \Omega) \text{ for every compact } K \in R^d, \quad (3.1)$$

$$D^\alpha u_n(\cdot + i_n T) \rightarrow u \text{ weakly in } L^2(\Omega), \quad |\alpha| = 2i - 1, \quad i = 1, 2, \dots, N. \quad (3.2)$$

The positivity of $[A_{\alpha\beta}]$ and Fatou's lemma imply

$$E(u) \leq \lim_{n \rightarrow \infty} E(u_n) = I,$$

so u is an energy minimizer. $\square$

Once we have an energy minimizer u , we know that u solves the corresponding Euler-Lagrange equation and the proof of theorem 3.1 is complete.

Corollary 3.5 *The convergence in (3.1) and (3.2) is indeed*

$$u_n(\cdot + i_n T) \rightarrow u \text{ in } C(\bar{\Omega}),$$

$$D^\alpha u_n(\cdot + i_n T) \rightarrow D^\alpha u \text{ strongly in } L^2(\Omega), \quad |\alpha| = 2i - 1, \quad i = 1, 2, \dots, N.$$

Proof. The uniform estimate of the $u_n(\cdot + i_n T)$'s in Ω_1 and Ω_2 gives the convergence in $C(\bar{\Omega})$. To see the strong convergence in $L^2(\Omega)$, one notes that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n dx \\ &= \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u D^\beta u dx. \end{aligned} \quad (3.3)$$

We then consider

$$\begin{aligned} & \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha (u_n - u) D^\beta (u_n - u) dx \\ &= \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n dx + \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u D^\beta u dx \\ & \quad - \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u dx - \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u D^\beta u_n dx. \end{aligned}$$

(3.2) implies

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n dx = \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u D^\beta u dx, \\ & \lim_{n \rightarrow \infty} \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u D^\beta u_n dx = \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u D^\beta u dx. \end{aligned}$$

Therefore with the help of (3.3) we deduce

$$\lim_{n \rightarrow \infty} \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^{\alpha}(u_n - u) D^{\beta}(u_n - u) dx = 0. \quad (3.4)$$

The positivity of $[A_{\alpha\beta}]$ implies that there exists $c > 0$ such that

$$\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^{\alpha}(u_n - u) D^{\beta}(u_n - u) \geq c \sum_{i=1}^N \sum_{|\alpha|=2i-1} |D^{\alpha}(u_n - u)|^2$$

With the help of (3.4) we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} \sum_{i=1}^N \sum_{|\alpha|=2i-1} |D^{2i-1}(u_n - u)|^2 = 0$$

which gives the strong convergence. $\square$

4 Proof of lemma 3.3

Suppose $\lambda \in (0, \Lambda)$. From statement III in section 2, we have for every small $\epsilon > 0$, numbers $r > 0$, and r_n , with $r_n \rightarrow \infty$ as $n \rightarrow \infty$, satisfying

$$\lambda - \epsilon < Q_n(r), Q_n(r_n) < \lambda + \epsilon$$

for large n . We can find y_n such that

$$\begin{aligned} \int_{B_{\Omega}(y_n, r)} W(u_n) dx &\in (\lambda - \epsilon, \lambda + \epsilon), \\ \int_{\Omega \setminus B_{\Omega}(y_n, r_n)} W(u_n) dx &\in (\Lambda - \lambda - 2\epsilon, \Lambda - \lambda + 2\epsilon) \end{aligned} \quad (4.1)$$

for large n . If we choose r large enough we can write

$$B_{\Omega}(y_n, r_n) \setminus \overline{B}_{\Omega}(y_n, r) = D_{1n} \cup D_{2n} \quad (4.2)$$

where D_{1n} and D_{2n} are two connected open sets. We also denote

$$\Omega \setminus \overline{B}_{\Omega}(y_n, r) = M_{1n} \cup M_{2n} \quad (4.3)$$

where M_{1n} and M_{2n} are again two connected open sets. We assume D_{1n} (D_{2n} , respectively) is contained in M_{1n} (M_{2n} , respectively) which has tail E_1 (E_2 , respectively). Clearly

$$\text{mes}(D_{1n}), \text{mes}(D_{2n}) \rightarrow \infty$$

where $\text{mes}(D)$ denotes the measure of D . For large n

$$\int_{D_{1n}} W(u_n)dx < 2\epsilon, \int_{D_{2n}} W(u_n)dx < 2\epsilon. \quad (4.4)$$

If we choose ϵ small enough, we see the u_n 's stay close to -1 or 1 in D_{1n} and D_{2n} in view of lemma 2.2. We will consider the three cases.

Case 1: Suppose, taking a subsequence if necessary, the u_n 's stay close to -1 in D_{2n} .

We will show that in this case the u_n 's already have large energies in M_{2n} so they can not constitute an energy minimizing sequence. Without loss of generality, we assume the period T of Ω is $T = (T^1, 0, \dots, 0)$. Let ξ be a non-negative smooth cut-off function such that

$$\begin{cases} \xi(t) = 0 & \text{if } t < 0, \\ \xi(t) = 1 & \text{if } t > 1, \end{cases} \quad (4.5)$$

$0 \leq \xi(t) \leq 1$, and $|D^k \xi(x)| \leq C_1$ for $k = 1, 2, \dots, 2N - 1$. Set

$$v_n(x) = (u_n(x) + 1)\xi\left(\frac{x^1 - (y_n^1 + (r + r_n)/2)}{h}\right) - 1 \quad (4.6)$$

where $x = (x^1, \dots, x^d)$ and $y_n = (y_n^1, \dots, y_n^d)$ and h is a large number to be specified later. Note $v_n \in \mathcal{A}$. We fix h and take n large enough so that $h < (r_n - r)/2$.

We will compare $\int_{M_{2n}} W(u_n)dx$ with $\int_{\Omega} W(v_n)dx$. Since for $x \in M_{2n}$, u_n and v_n only differ in D_{2n} , and on that set v_n is closer to -1 than u_n is, so we have

$$\int_{\Omega} W(v_n)dx \leq \int_{M_{2n}} W(u_n)dx. \quad (4.7)$$

We also compare

$$\int_{M_{2n}} A_{\alpha\beta} D^{\alpha} u_n D^{\beta} u_n dx \text{ with } \int_{\Omega} A_{\alpha\beta} D^{\alpha} v_n D^{\beta} v_n dx.$$

Since the two quantities agree in $M_{2n} \setminus D_{2n}$ and $D^{\alpha} v_n$ vanishes in $\Omega \setminus M_{2n}$, we need only to compare

$$\int_{D_{2n}} A_{\alpha\beta} D^{\alpha} u_n D^{\beta} u_n dx \text{ with } \int_{D_{2n}} A_{\alpha\beta} D^{\alpha} v_n D^{\beta} v_n dx.$$

Consider

$$\begin{aligned} & A_{\alpha\beta} D^{\alpha} v_n(x) D^{\beta} v_n(x) = \\ & A_{\alpha\beta} \left[\sum_{\iota \leq \alpha} \frac{\alpha!}{\iota!} D^{\alpha-\iota} (u_n(x) + 1) D^{\iota} \xi\left(\frac{x^1 - (y_n^1 + (r + r_n)/2)}{h}\right) \right] \end{aligned}$$

$$\begin{aligned}
& \times \left[\sum_{\kappa \leq \beta} \frac{\beta!}{\kappa!} D^{\beta-\kappa} (u_n(x) + 1) D^\kappa \xi \left(\frac{x^1 - (y_n^1 + (r + r_n)/2)}{h} \right) \right] \\
& = A_{\alpha\beta} \left[\sum_{\iota_1 \leq \alpha_1} \frac{\alpha!}{\iota_1!} D^{\alpha-(\iota_1, 0, \dots, 0)} (u_n(x) + 1) D^{\iota_1} \xi \left(\frac{x^1 - (y_n^1 + (r + r_n)/2)}{h} \right) \right] \\
& \times \left[\sum_{\kappa_1 \leq \beta_1} \frac{\beta!}{\kappa_1!} D^{\beta-(\kappa_1, 0, \dots, 0)} (u_n(x) + 1) D^{\kappa_1} \xi \left(\frac{x^1 - (y_n^1 + (r + r_n)/2)}{h} \right) \right].
\end{aligned}$$

The first term from the product is

$$A_{\alpha\beta} \xi^2 \left(\frac{x^1 - (y_n^1 + (r + r_n)/2)}{h} \right) D^\alpha u_n D^\beta u_n \quad (4.8)$$

and the other terms assume the form

$$\begin{aligned}
& \frac{A_{\alpha\beta} \alpha! \beta!}{h^{\iota_1 + \kappa_1} \iota_1! \kappa_1!} D^{\alpha-(\iota_1, 0, \dots, 0)} (u_n(x) + 1) D^{\beta-(\kappa_1, 0, \dots, 0)} (u_n(x) + 1) \\
& \xi^{(\iota_1)} \left(\frac{x^1 - (y_n^1 + (r + r_n)/2)}{h} \right) \xi^{(\kappa_1)} \left(\frac{x^1 - (y_n^1 + (r + r_n)/2)}{h} \right)
\end{aligned}$$

where at least one of ι_1 and κ_1 is not equal to 0. They can be bounded by

$$\frac{C}{h^{\iota_1 + \kappa_1}} |D^{\alpha-(\iota_1, 0, \dots, 0)} (u_n(x) + 1) D^{\beta-(\kappa_1, 0, \dots, 0)} (u_n(x) + 1)|.$$

Lemma 4.1 *With u_n as above, there is $C > 0$ and a positive integer n_1 so that for $n \geq n_1$*

$$\int_{D_{2n}} \frac{1}{h^{\iota_1 + \kappa_1}} |D^{\alpha-(\iota_1, 0, \dots, 0)} (u_n(x) + 1) D^{\beta-(\kappa_1, 0, \dots, 0)} (u_n(x) + 1)| dx \leq \frac{C}{h}$$

for all α, β , $|\alpha| = 2i - 1$, $|\beta| = 2j - 1$, $i, j \in \{1, 2, \dots, N\}$, $\iota_1 \in \{1, 2, \dots, 2i - 1\}$, $\kappa_1 \in \{1, 2, \dots, 2j - 1\}$, $\iota_1 + \kappa_1 \geq 1$.

Proof. In view of lemma 2.2, we know $u_n + 1$ is close to 0 uniformly for $x \in D_{2n}$. According to our assumption $W''(-1) > 0$, we find

$$(u_n(x) + 1)^2 \leq CW(u_n(x))$$

for $x \in D_{2n}$ where C is independent of large n . We then obtain

$$\int_{D_{2n}} (u_n(x) + 1)^2 dx \leq C$$

where C again is independent of large n . We also know that from the positivity of $[A_{\alpha\beta}]$

$$\int_{\Omega} [D^\alpha u_n(x)]^2 dx \leq C, \quad |\alpha| = 2i - 1, \quad i = 1, 2, \dots, N.$$

Appealing to (2.3) again, we deduce

$$\int_{\Omega} [D^{\alpha} u_n(x)]^2 dx \leq C, \quad |\alpha| = 2i, \quad i = 1, 2, \dots, N-1.$$

Lemma 4.1 follows from these two bounds and Holder's inequality. $\square$

We sum over α, β and find

$$\begin{aligned} \int_{D_{2n}} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^{\alpha} v_n D^{\beta} v_n dx &\leq \int_{D_{2n}} \xi^2 \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^{\alpha} u_n D^{\beta} u_n dx \\ &+ O\left(\frac{1}{h}\right) \leq \int_{D_{2n}} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^{\alpha} u_n D^{\beta} u_n dx + O\left(\frac{1}{h}\right). \end{aligned}$$

Together with (4.7) and the fact $v_n \in \mathcal{A}$, we deduce

$$\begin{aligned} I \leq E(v_n) &= \int_{\Omega} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^{\alpha} v_n D^{\beta} v_n + W(v_n) \right] dx \\ &\leq \int_{M_{2n}} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^{\alpha} u_n D^{\beta} u_n + W(u_n) \right] dx + O\left(\frac{1}{h}\right). \end{aligned}$$

If we choose h large enough, we obtain

$$\begin{aligned} E(u_n) &\geq \int_{\Omega \setminus M_{2n}} W(u_n) dx + \int_{M_{2n}} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^{\alpha} u_n D^{\beta} u_n + W(u_n) \right] dx \\ &\geq \lambda + I - 3\epsilon \end{aligned}$$

which implies that $\{u_n\}$ can not be an energy minimizing sequence.

Case 2: Suppose, taking a subsequence if necessary, the u_n 's stay close to 1 in D_{1n} .

This possibility can be ruled out by the same argument as in the previous case. Instead of considering D_{2n} and M_{2n} , one considers D_{1n} and M_{1n} . One takes

$$v_n(x) = (u_n(x) - 1) \xi\left(\frac{-x^1 + (y_n^1 - (r + r_n)/2)}{h}\right) + 1. \quad (4.9)$$

Case 3: Suppose, taking a subsequence if necessary, the u_n 's stay close to -1 in D_{1n} and 1 in D_{2n} .

We truncate u_n outside $B_\Omega(y_n, r_n)$ and replace it by v_n . These have approximately equal energies being at least close to I there, but the two tails of u_n have extra energies bounded away from 0. Therefore such u_n 's can not constitute an energy minimizing sequence. Set

$$w_n = \{[(u_n(x) + 1)\xi(\frac{x^1 - (y_n^1 - (r + r_n)/2)}{h}) - 1] - 1\} \\ \times \xi(\frac{-x^1 + (y_n^1 + (r + r_n)/2)}{h}) + 1 \quad (4.10)$$

where $h < (r_n - r)/2$ is a large number to be specified later.

We first compare $\int_{B_\Omega(y_n, r_n)} W(u_n)dx$ with $\int_\Omega W(w_n)dx$. Observe that u_n and w_n only differ in $M_{1n} \cup M_{2n}$ for large n , and outside $B_\Omega(y_n, r_n)$, $W(w_n) = 0$. Also in D_{1n} , where u_n is close to -1, w_n is closer to -1, and in D_{2n} , where u_n is close to 1, w_n is closer to 1. Therefore we deduce

$$\int_\Omega W(w_n)dx \leq \int_{B_\Omega(y_n, r_n)} W(u_n)dx. \quad (4.11)$$

We then compare

$$\int_{B_\Omega(y_n, r_n)} A_{\alpha\beta} D^\alpha u_n D^\beta u_n dx \text{ with } \int_\Omega A_{\alpha\beta} D^\alpha w_n D^\beta w_n dx.$$

We consider the integrals on three different sets, D_{1n} , D_{2n} and $B_\Omega(y_n, r)$. Since u_n and w_n agree in $B_\Omega(y_n, r)$ for large n . We only consider D_{1n} and D_{2n} .

In D_{1n} ,

$$\xi(\frac{x^1 + (y_n^1 + (r + r_n)/2)}{h}) = 1$$

for large n . Therefore

$$w_n(x) = [u_n(x) + 1]\xi(\frac{x^1 - (y_n^1 - (r + r_n)/2)}{d}) - 1.$$

In D_{2n} ,

$$\xi(\frac{x^1 - (y_n^1 - (r + r_n)/2)}{h}) = 1$$

for large n . Therefore

$$w_n(x) = [u_n(x) - 1]\xi(\frac{-x^1 + (y_n^1 + (r + r_n)/2)}{h}) + 1.$$

Using the same argument as the one in case 1, we find

$$\int_\Omega \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha w_n D^\beta w_n dx$$

$$\leq \int_{B\Omega(y_n, r_n)} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n dx + O\left(\frac{1}{h}\right).$$

Together with (4.11) and the fact $w_n \in \mathcal{A}$ we deduce

$$\begin{aligned} I &\leq \int_{\Omega} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^{2i-1} w_n D^\alpha w_n + W(w_n) \right] dx \\ &\leq \int_{B\Omega(y_n, r_n)} \left[\sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n + W(u_n) \right] dx + O\left(\frac{1}{h}\right). \end{aligned}$$

If we choose h large enough, we obtain

$$\begin{aligned} E(u_n) &\geq \int_{\Omega \setminus B\Omega(y_n, r_n)} W(u_n) dx \\ &+ \int_{B\Omega(y_n, r_n)} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n + W(u_n) \right] dx \geq \Lambda - \lambda + I - 3\epsilon \end{aligned}$$

which implies that $\{u_n\}$ can not be an energy minimizing sequence if we choose ϵ small. This rules out the third possibility and proves lemma 3.3.

5 A sufficient condition for non-periodic Ω

In this section we will study problem (1.1) in a non-periodic tube. The lack of periodicity causes the loss of translation. This brings a subtle difference between problem (1.1) in a periodic tube and that in a non-periodic tube. We will only consider non-periodic tubes that are “half-periodic” near the two ends. In this case, the concentration-compactness principle can give a sufficient condition which ensures the existence of a transition layer solution.

Let Ω be an open connected set in R^d . Suppose Ω can be divided into three mutually disjoint connected open sets Ω_1 , Ω_2 and Ω_0 where $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_0}$. We further assume that Ω_0 is bounded and Ω_1 and Ω_2 are “half periodic”. That Ω_1 is half periodic means there exists $T_1 \in R^d$ such that for every $x_1 \in \Omega_1$, $x_1 + T_1 \in \Omega_1$, and for every $x_1 \in \Omega_1$ there exist $y_1 \in \Omega_0$ and a non-negative integer n_1 satisfying $x_1 = y_1 + n_1 T_1$. The same condition applies to Ω_2 . Ω_1 and Ω_2 are indeed the two half periodic tails of Ω . We associate the end E_1 to Ω_1 and the other end E_2 to Ω_2 (see figure 2).

The function class $\mathcal{A}$ is defined in (2.5). For $u \in \mathcal{A}$ we require

$$\lim_{x \rightarrow E_1} u(x) = -1, \quad \lim_{x \rightarrow E_2} u(x) = 1.$$

Figure 2: An example of non-periodic tubes.

We define G_1 (G_2 , respectively) to be the “periodic extension” of Ω_1 (Ω_2 , respectively).

$$G_1 = \{x \in R^d : x + n_1 T_1 \in \Omega_1 \text{ for some integer } n_1\}. \quad (5.1)$$

We require that G_1 (G_2 , respectively) has bounded cross sections perpendicular to T_1 (T_2 , respectively, where T_2 is the period of G_2 .) We denote the other end of G_1 (G_2 , respectively) by E'_1 (E'_2 , respectively) (see figure 2).

We now define two auxiliary variational problems:

$$\begin{aligned} I_1 &= \inf_{\mathcal{A}_1} \int_{G_1} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u D^\beta u + W(u) \right] dx, \\ I_2 &= \inf_{\mathcal{A}_2} \int_{G_2} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u D^\beta u + W(u) \right] dx. \end{aligned} \quad (5.2)$$

The function classes $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined in a way similar to (2.5). We require for $u \in \mathcal{A}_1$, $\lim_{x \rightarrow E_1} u(x) = -1$, $\lim_{x \rightarrow E'_1} u(x) = 1$, and for $u \in \mathcal{A}_2$, $\lim_{x \rightarrow E'_2} u(x) = -1$, $\lim_{x \rightarrow E_2} u(x) = 1$.

Theorem 5.1 *Let Ω be a tube half-periodic near the two ends. We always have*

$$I \leq \min\{I_1, I_2\}$$

where I is defined in (1.1) and I_1, I_2 are defined in (5.2). If the strict inequality

$$I < \min\{I_1, I_2\}$$

holds, then problem (1.1) has an energy minimizer in $\mathcal{A}$.

Proof. Without the loss of generality, we assume $I_1 \leq I_2$. We know from theorem 3.1 that the infimum I_1 can be achieved by a function, say v , in $\mathcal{A}_1$. We assume without losing the generality $T_1 = (T_1^1, 0, \dots, 0)$, $T_1^1 < 0$ and $(0, 0, \dots, 0) \in \Omega_1$. Set

$$v_n(x) = \begin{cases} [v(x + nT_1) - 1] \xi\left(\frac{-x^1}{h}\right) + 1, & x \in \Omega_1, \\ 1, & x \in \Omega \setminus \Omega_1 \end{cases} \quad (5.3)$$

where ξ is defined in (4.5) and n is a positive integer. Note $v_n \in \mathcal{A}$. We compare the energy of v_n in $\mathcal{A}$ with that of v in $\mathcal{A}_1$. When n is large, v_n is close to 1 for $x^1 \in [-h, 0]$, so

$$\int_{\Omega} W(v_n) dx = \int_{\substack{x \in \Omega \\ x^1 < 0}} W(v_n) \leq \int_{G_1} W(v) dx. \quad (5.4)$$

Consider

$$A_{\alpha\beta} D^\alpha v_n D^\beta v_n =$$

$$\begin{aligned}
& A_{\alpha\beta} \left[\sum_{\iota_1 \leq \alpha_1} \frac{\alpha!}{\iota_1!} D^{\alpha-(\iota_1,0,\dots,0)} (v(x+nT_1)-1) D^{\iota_1} \xi\left(\frac{-x^1}{h}\right) \right] \\
& \times \left[\sum_{\kappa_1 \leq \beta_1} \frac{\beta!}{\kappa_1!} D^{\beta-(\kappa_1,0,\dots,0)} (v(x+nT_1)-1) D^{\kappa_1} \xi\left(\frac{-x^1}{h}\right) \right].
\end{aligned}$$

The first term from the product is

$$A_{\alpha\beta} D^\alpha v(x+nT_1) D^\beta v(x+nT_1) \xi^2\left(\frac{-x^1}{h}\right)$$

and the other terms assume the form

$$\begin{aligned}
& \frac{A_{\alpha\beta} \alpha! \beta!}{h^{\iota_1+\kappa_1} \iota_1! \kappa_1!} D^{\alpha-(\iota_1,\dots,0)} v(x+nT_1) D^{\beta-(\kappa_1,0,\dots,0)} v(x+nT_1) \\
& \times \xi^{(\iota_1)}\left(\frac{-x^1}{h}\right) \xi^{(\kappa_1)}\left(\frac{-x^1}{h}\right)
\end{aligned}$$

where $\iota_1 + \kappa_1 \geq 1$. We sum over α, β and integrate over Ω . We find, as in the proof of lemma 3.3,

$$\begin{aligned}
& \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} A_{\alpha\beta} D^\alpha v_n D^\beta v_n dx \\
& = \int_{G_1 \cap \{x^1 > -1-nT_1^1\}} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha v D^\beta v dx + O\left(\frac{1}{h}\right).
\end{aligned}$$

Together with (5.4), we deduce

$$\begin{aligned}
I & \leq \int_{\Omega} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} A_{\alpha\beta} D^\alpha v_n D^\beta v_n + W(v_n) \right] dx \\
& \leq \int_{G_1 \cap \{x^1 > -1-nT_1^1\}} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha v D^\beta v dx + \int_{G_1} W(v) dx + O\left(\frac{1}{h}\right).
\end{aligned}$$

If we choose h comparable with n and send $n \rightarrow \infty$, we find

$$I \leq \int_{G_1} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha v D^\beta v + W(v) \right] dx = I_1.$$

This proves the first part of theorem 5.1.

We now assume $I < \min\{I_1, I_2\}$. We again consider the concentration functions, the Q_n 's and Q , of the $W(u_n)$'s where $\{u_n\}$ is an energy minimizing sequence in $\mathcal{A}$. We consider the three cases $\lambda = 0$, $\lambda \in (0, \Lambda)$ and $\lambda = \Lambda$. As in

the proof of theorem 3.1, we can rule out the cases $\lambda = 0$ and $\lambda \in (0, \Lambda)$. We are left with the case $\lambda = \Lambda$ which implies from statement I in section 2 that there exist a subsequence of u_n , again denoted by u_n , and $y_n \in \Omega$ such that for every $\epsilon > 0$ there exists $r > 0$ so that

$$\int_{B_\Omega(y_n, r)} W(u_n(x)) dx \geq \Lambda - \epsilon$$

for large n .

We will prove that the y_n 's indeed stay in a bounded subset of Ω . Suppose the contrary, so we can assume a subsequence of $\{y_n\}$, still denoted by $\{y_n\}$ approaches E_1 . Without the loss of generality, we also assume $(0, 0, \dots, 0) \in \Omega_1$, $T_1 = (T_1^1, 0, \dots, 0)$, $T_1^1 < 0$. Define in G_1

$$v_n(x) = \begin{cases} (u_n(x) - 1)\xi(\frac{-x^1}{h}) + 1, & x \in \Omega_1 \\ 1, & x \in G_1 \setminus \Omega_1. \end{cases} \quad (5.5)$$

Here we first choose $h < -y_n^1 - r$.

We observe since the u_n 's are close to 1 in $\Omega_0 \cup \Omega_2$,

$$\int_{G_1} W(v_n) dx = \int_{\Omega_1} W(v_n) dx \leq \int_{\Omega} W(u_n) dx. \quad (5.6)$$

As in the proof of lemma 3.3, we write

$$\begin{aligned} A_{\alpha\beta} D^\alpha v_n(x) D^\beta v_n(x) &= A_{\alpha\beta} D^\alpha [(u_n(x) - 1)\xi(\frac{-x^1}{h})] D^\beta [(u_n(x) - 1)\xi(\frac{-x^1}{h})] \\ &= A_{\alpha\beta} \left[\sum_{\iota_1 \leq \alpha_1} \frac{\alpha!}{\iota_1!} D^{\alpha - (\iota_1, 0, \dots, 0)} (u_n(x) - 1) D^{\iota_1} \xi(\frac{-x^1}{h}) \right] \\ &\quad \times \left[\sum_{\kappa_1 \leq \beta_1} \frac{\beta!}{\kappa_1!} D^{\beta - (\kappa_1, 0, \dots, 0)} (u_n(x) - 1) D^{\kappa_1} \xi(\frac{-x^1}{h}) \right] \end{aligned}$$

where the first term from the product is

$$A_{\alpha\beta} \xi^2(\frac{-x^1}{h}) D^\alpha u_n D^\beta u_n$$

and the other terms assume the form

$$\frac{A_{\alpha\beta} \alpha! \beta!}{h^{\iota_1 + \kappa_1} \iota_1! \kappa_1!} D^{\alpha - (\iota_1, 0, \dots, 0)} (u_n - 1) D^{\beta - (\kappa_1, 0, \dots, 0)} (u_n - 1) \xi^{(\iota_1)}(\frac{-x^1}{h}) \xi^{(\kappa_1)}(\frac{-x^1}{h}),$$

$\iota_1 + \kappa_1 \geq 1$. Summing over α, β and integrating over G_1 , we find, as in the proof of lemma 3.3,

$$\int_{G_1} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha v_n D^\beta v_n dx = \int_{\Omega_1} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha v_n D^\beta v_n dx$$

$$\begin{aligned}
&\leq \int_{\Omega_1} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n dx + O\left(\frac{1}{h}\right) \\
&\leq \int_{\Omega} \sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n dx + O\left(\frac{1}{h}\right).
\end{aligned}$$

Together with (5.5), we deduce

$$\begin{aligned}
I_1 &\leq \int_{G_1} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha v_n D^\beta v_n + W(v_n) \right] dx \\
&\leq \int_{\Omega} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_n D^\beta u_n + W(u_n) \right] dx + O\left(\frac{1}{h}\right) = I + O\left(\frac{1}{h}\right).
\end{aligned}$$

If we choose h comparable with $-y_n^1$ and send $n \rightarrow \infty$, we obtain $I_1 \leq I$ which contradicts the assumption $I < \min\{I_1, I_2\}$.

Once we know the y_n 's stay in a bounded subset of Ω , we conclude that there exists $u \in \mathcal{A}$ such that passing to a subsequence if necessary

$$u_n \rightarrow u \text{ in } C(K) \text{ for every compact } K \subset \overline{\Omega}, \quad (5.7)$$

$$D^\alpha u_n \rightarrow D^\alpha u \text{ weakly in } L^2(\Omega), \quad |\alpha| = 2i - 1, \quad i = 1, 2, \dots, N. \quad (5.8)$$

As in the proof of lemma 3.4, this u is an energy minimizer of (1.1). $\square$

Similar to corollary 3.5, we have

Corollary 5.2 *If $I < \min\{I_1, I_2\}$, the convergence in (5.7) and (5.8) is indeed*

$$u_n \rightarrow u \text{ in } C(\overline{\Omega}),$$

$$D^\alpha u_n \rightarrow D^\alpha u \text{ strongly in } L^2(\Omega), \quad |\alpha| = 2i - 1, \quad i = 1, 2, \dots, N.$$

6 Some applications of theorem 5.1

We will construct a few examples of Ω to illustrate theorem 5.1 in this section. We assume that equation (1.5) has the property of unique continuation, namely if a solution u of (1.5) is equal to a constant in an open subset of Ω , then u is necessarily a constant in Ω . If W is real analytic, then the solutions of (1.5) are real analytic and (1.5) in turn has the property of unique continuation.

Our first example is a near periodic tube Ω_A in R^2 . The corresponding periodic extensions G_1, G_2 of the tails are congruent to G_A (see figure 3). Note that the cross sections of Ω_A have the same lengths as those of the corresponding cross sections of G_A except in a middle region where the cross sections of Ω_A have shorter lengths than that of G_A .

Figure 3: The picture of Ω_A .

Let u_A be an energy minimizer of problem (1.1) in G_A . The existence of such an energy minimizer is guaranteed by theorem 3.1. Denote this problem by

$$I_G = \inf_{u \in \mathcal{A}_G} \int_{G_A} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u D^\beta u + W(u) \right] dx. \quad (6.1)$$

Clearly $I_G = I_1 = I_2$ where I_1 and I_2 are the infima of the corresponding auxiliary problems. Take u_A to be a test function and put it in Ω_A . Because the middle region of Ω_A is thinner than that of G_A , we have

$$\begin{aligned} I &\leq \int_{\Omega_A} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_A D^\beta u_A + W(u_A) \right] dx \\ &< \int_{G_A} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_A D^\beta u_A + W(u_A) \right] dx = I_G. \end{aligned} \quad (6.2)$$

The strict inequality in (6.2) follows from the unique continuation which ensures that u_A is not a constant in any open subset of G_A . Therefore in this tube there is an energy minimizer.

The second example is a near perfect tube except that somewhere in the middle there is a hole, but the lengths of the cross sections are identical. We denote the tube, its tails and their congruent periodic extension by Ω_B , $\Omega_{B,1}$,

Figure 4: The picture of Ω_B .

$\Omega_{B,2}$ and G_B (see figure 4). We take u_B to be an energy minimizer of (1.1) in G_B . It is not clear to us whether u_B is necessarily one-dimensional, i.e., u_B is a function of x_1 only. If we are willing to make the assumption that u_B is one-dimensional, then we extend u_B to all R^2 in the obvious way and put it in Ω_B . Because the cross sections of Ω have the same length, we have

$$\begin{aligned} & \int_{\Omega_B} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_B D^\beta u_B + W(u_B) \right] dx \\ &= \int_{G_B} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_B D^\beta u_B + W(u_B) \right] dx = I_G. \end{aligned}$$

However u_B is not an energy minimizer in Ω_B since it does not satisfy the boundary conditions of the Euler-Lagrange equation (1.5) in Ω_B . Therefore

$$I < \int_{\Omega_B} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} A_{\alpha\beta} D^\alpha u_B D^\beta u_B + W(u_B) \right] dx = I_G \quad (6.3)$$

which again yields an energy minimizer in Ω_B .

Figure 5: The picture of Ω_C .

The third example will show that the sufficient condition is not always valid. Indeed in this example, problem (1.5) does not have an energy minimizer. Let Ω_C be a near perfect tube. Unlike the first example, we assume that somewhere in the tube the cross sections have longer lengths than that of the corresponding G_C (see figure 5). If u were an energy minimizer of (1.1) in Ω_C , then we would have

$$\begin{aligned} I &= \int_{\Omega_C} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} D^\alpha u D^\beta u + W(u) \right] dx \\ &> \int_{G_C} \left[\sum_{i,j=1}^N \sum_{\substack{|\alpha|=2i-1 \\ |\beta|=2j-1}} D^\alpha u D^\beta u + W(u) \right] dx \geq I_G = I_1 = I_2. \end{aligned}$$

This contradicts the first part of theorem 5.1.

Note added in proof. In a forthcoming paper, [11], the second named author is able to prove theorems 3.1 and 5.1 without condition (1.4).

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