The Geometry of Fox's Free Calculus with Applications to Higher Dimensional Knots

Work in Progress

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PowerPoint slides can be found at:
www.csee.umbc.edu/~lomonaco/Lectures.html

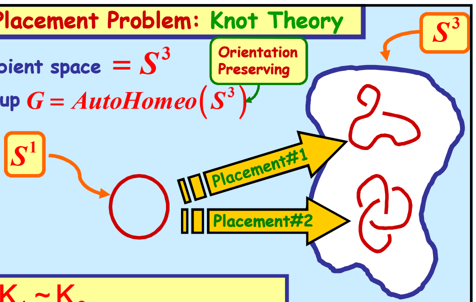
Road Map for Talk

- 3-D Knot Theory
  - Knot Diagrams, Reidemeister Moves, π_1 ,Wirtinger
  - Geometry of Group Presentations
  - Wirtinger Generalized Presentations
  - Geometry of Fox Free Calc
- 4-D Knot Theory
  - History + Asphericity Analogs
  - Cross Sections & Midsections
  - Yoshikawa Moves and Swenton Completeness
  - Generalized Presentations of 2-Knots ???
- Conclusion

Knot Theory

Placement Problem: Knot Theory

- Ambient space $= S^3$
- Group $G = AutoHomeo\left(S^3\right)$



Def. $K_1 \sim K_2$
if $g \in G$ s.t. $gK_1 = K_2$

Problem. When are two placements the same ?
$K_1 \sim K_2$?

Equivalent Definition

Def. $K_1 \sim K_2$ provided there exists a continuous family of auto-homeomorphisms

$$h_t : S^3 \to S^3 \quad (0 \le t \le 1)$$

i.e., an isotopy, that continuously deforms K_1 into K_2.

Various Placement Problems

- 3-D Knot Theory

  $k : S^1 \to S^3$ 1-Knot (S^3, kS^1)

- 4-D Knot Theory

  $k : S^2 \to S^4$ 2-Knot (S^4, kS^2)

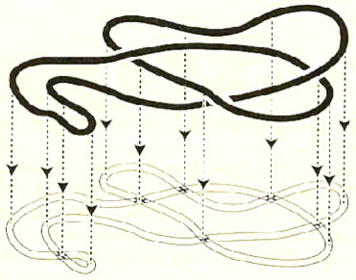
- 5-D Knot Theory

  $k : S^3 \to S^5$ 3-Knot (S^5, kS^3)

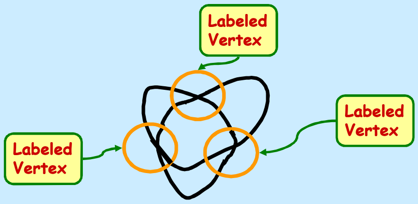
3-D Knot Theory

1-Knots

Knot Projections

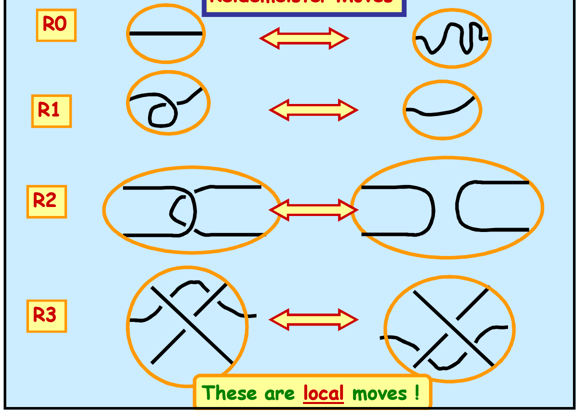


Knot Diagram



- Planar four valent graph with
- Labeled vertices

Reidemeister Moves



These are local moves !

When do two Knot diagrams represent the same or different knots ?

Theorem (Reidemeister). Two knot diagrams represent the same knot iff one can be transformed into the other by a finite sequence of Reidemester moves.

What is a knot invariant ?

Def. A knot invariant **I** is a map

$$I : Knots \to Mathematical\ Domain$$

that takes each knot **K** to a mathematical object **I(K)** such that

$$K_1 \sim K_2 \Rightarrow I(K_1) = I(K_2)$$

Consequently,

$$I(K_1) \neq I(K_2) \Rightarrow K_1 \not\sim K_2$$

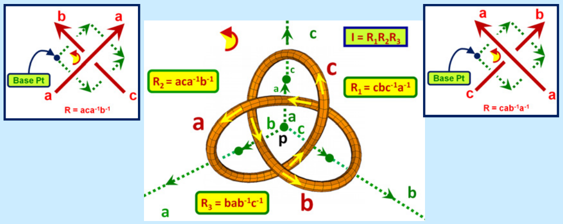
The Fundamental Group $\pi_1(X)$

Knot Exterior $X = S^3 - SmallOpenTubularNbd\left(kS^1\right)$

- Fundamental Group $\pi_1(X) =$ Knot Invariant
- Asphericity of Knots (Papakyriakopoulos)

$$\Rightarrow X = K(\pi_1 X, 1). \quad \therefore \pi_n X = 0 \ for\ n > 1$$

The Wirtinger Presentation



$$\left(\underbrace{a,b,c}_{\text{Generators}} : \underbrace{r_1,r_2,r_3}_{\text{Relators}}\right)$$

Group Presentations

Question: When do two presentations represent the same group ?

Tietze Transformations: $T_1^{\pm 1}, T_2^{\pm 1}$

Tietze 1: $\left(\underline{\underline{x}} : \underline{\underline{r}}\right) \xrightarrow{T_1} \left(\underline{\underline{x}} \cup y : \underline{\underline{r}} \cup s\right)$, where

- y is a new symbol, and
- $s = y\xi^{-1}$, with $\xi \in F\left(\underline{\underline{x}}\right)$

Tietze 2: $\left(\underline{\underline{x}} : \underline{\underline{r}}\right) \xrightarrow{T_2} \left(\underline{\underline{x}} : \underline{\underline{r}} \cup s\right)$, where

- $s \in Cons\left(\underline{\underline{r}}\right)$, i.e., $s = \prod_{\alpha=1}^{m} r_{j(\alpha)}^{w_\alpha}$, with
- $w_\alpha \in F\left(\underline{\underline{x}}\right)$, $0 \leq \alpha \leq m$, and
- $r_{j(\alpha)}^{w_\alpha} = w_\alpha \cdot r_{j(\alpha)} \cdot w_\alpha^{-1}$

Group Presentations

Question: When do two presentations represent the same group ?

Theorem (Tietze): Two group presentations represent the same group iff there exists a finite sequence of Tietze transformations which transforms one into the other.

The Geometry of Group Presentations

Def. An abstract Group presentation consists of two sets

- $\underline{\underline{x}}$, the set of generators,
- $\underline{\underline{r}}$, the set of relators,

together with an evaluation map

$$\hat{} : \underline{\underline{r}} \to F\left(\underline{\underline{x}}\right)$$

from the set of relators $\underline{\underline{r}}$ into the free group $F\left(\underline{\underline{x}}\right)$ on the set of symbols $\underline{\underline{x}}$.

The Geometry of Group Presentations

Example: $\left(a,b,c : r_1,r_2,r_3\right)$, where

$$\begin{cases} \hat{r}_1 = cbc^{-1}a^{-1} \\ \hat{r}_2 = aca^{-1}b^{-1} \\ \hat{r}_3 = bab^{-1}c^{-1} \end{cases}$$

The Geometry of Group Presentations

Def. A CW-complex is said to be monopointed it it has only one 0-cell.

Proposition. Up to renaming and reordering, there exists a one-to-one correspondence between the set of abstract group presentations and a set of monopointed 2-D CW-complexes.

$$\left(\underline{\underline{x}} : \underline{\underline{r}}\right) \leftrightarrow K\left(\underline{\underline{x}} : \underline{\underline{r}}\right)$$

The Geometry of Group Presentations

The CW-complex $K\left(\underline{\underline{x}} : \underline{\underline{y}}\right)$ is constructed with an initial 0-cell, denoted by ∞ , and then iteratively attaching cells as follows:

1-cells: For each generator $x_j \in \underline{\underline{x}}$, adjoin an oriented 1-cell X_j by attaching both endpoints to the sole 0-cell ∞ .

2-cells: For each relator $r_k \in \underline{\underline{r}}$, adjoin an oriented 2-cell R_k with attaching map $\hat{r}_k$.

Examples

Example: $Torus = \left(a,b : r\right), \hat{r} = ab\bar{a}\bar{b}$

Example: $\mathbb{R}P^2 = \left(a,b : r\right), \hat{r} = aba\bar{b}$

Example: $S^1 \vee D^2 = \left(a,b : r\right), \hat{r} = b$

Example: $S^1 \vee S^1 \vee S^2 = \left(a,b : r\right), \hat{r} = 1$

Example: $S^2 \vee S^2 = \left(\varnothing : r_1,r_2\right), \hat{r}_1 = 1, \hat{r}_2 = 1$

The Geometry of the Tietze Moves

- Tietze 1 attaches a 2-cell S and a free edge Y .

$$T_1 : \left(\underline{\underline{x}} : \underline{\underline{r}}\right) \mapsto \left(\underline{\underline{x}} \cup y : \underline{\underline{r}} \cup s\right), \hat{s} = y\xi^{-1}, \xi \in F\left(\underline{\underline{x}}\right)$$

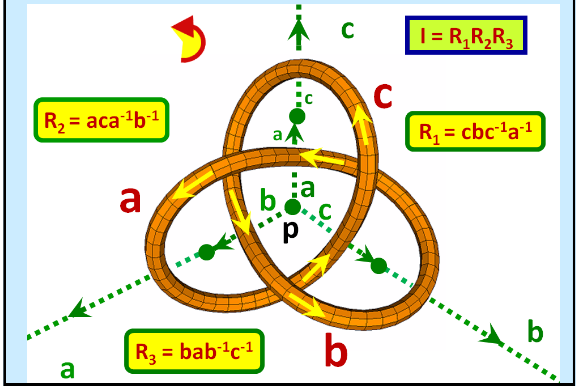
Thus, T_1 is a simple homotopy operation; and therefore preserves homotopy type.

- Tietze 2 attaches a 2-cell S.

$$T_2 : \left(\underline{\underline{x}} : \underline{\underline{r}}\right) \mapsto \left(\underline{\underline{x}} : \underline{\underline{r}} \cup s\right), \hat{s} \in Cons\left(\underline{\underline{r}}\right)$$

Thus, T_2 does <u>NOT</u> necessarily preserve homotopy type !

The Geometry of the Wirtinger Presentation



A Paradox ?

<u>Paradox</u>: The Wirtinger presentation actually points to a 3-D CW decomposition of the exteror X. But it only represents a 2-D CW-subcomplex !

<u>The</u> <u>usual</u> "<u>fix</u>": The Wirtinger presentation has one too many generators. So toss out an unnecessary relator by applying a Tietze 2 move.

Fortunately, in this particular case, the Tietze 2 move preserves the homotopy type because there is a simpe homotopy type move on the 3-D complex collapsing the 3-D complex to the same resuting 2-D complex

Observation

The 3-cell in X corresponds to an <u>identity</u> I among the relators, namely:

$$I \xrightarrow{\wedge} \hat{I} = r_1 r_2 r_3 \xrightarrow{\wedge} cb\bar{c}\bar{a} \cdot ac\bar{a}\bar{b} \cdot ba\bar{b}\bar{c} = 1$$

Where does this identity "live" ?

Groups with Operators

<u>Def</u>. Let H and G be groups. The group G is said to be an H-<u>group</u> provided there exists a morphism $H \to Aut(G)$ of H into the group Aut(G) of automorpisms of G .

Free $F\left(\underline{\underline{x}}\right)$ -groups

<u>Def</u>. Let $\underline{\underline{x}}$ and $\underline{\underline{t}}$ be two disjoint sets of symbols, and let $F\left(\underline{\underline{x}}\right)$ and $F\left(\underline{\underline{x}} \cup \underline{\underline{t}}\right)$ denote the corresponding free groups, respectively. The <u>free</u> $F\left(\underline{\underline{x}}\right)$-<u>group</u> on the symbols $\underline{\underline{t}}$, written $\mathfrak{F}_{F\left(\underline{\underline{x}}\right)}\left(\underline{\underline{t}}\right)$, is the smallest normal subgroup of $F\left(\underline{\underline{x}} \cup \underline{\underline{t}}\right)$ containing $\underline{\underline{t}}$.

It immediately follows that $\mathfrak{F}_{F\left(\underline{\underline{x}}\right)}\left(\underline{\underline{t}}\right)$ is invariant under the conjugation action of $F\left(\underline{\underline{x}}\right)$.

Free $F(\underline{\underline{x}})$-groups

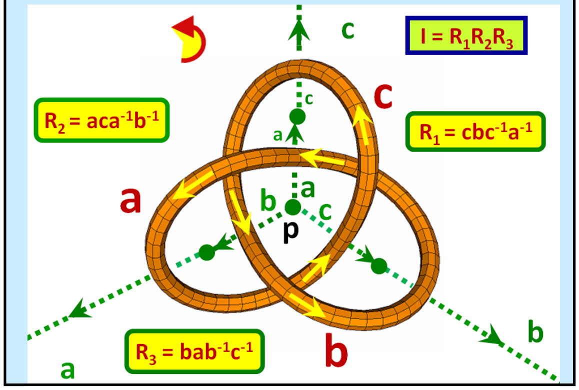
Thus, the elements of $\mathfrak{F}_{F(\underline{\underline{x}})}(\underline{\underline{t}})$ are of the form:

$$\prod_\alpha t_{j(\alpha)}^{w_\alpha}$$

where $t_{j(\alpha)}^{w_\alpha} = w_\alpha \cdot t_{j(\alpha)} \cdot w_{j(\alpha)}^{-1}$

Note. This conjugation action is a left action.

The Geometry of the Wirtinger Presentation



I = $R_1R_2R_3$

R_2 = aca^{-1}b^{-1}

R_1 = cbc^{-1}a^{-1}

R_3 = bab^{-1}c^{-1}

Wirtinger Generalized Presentation

The Wirtinger 3-D CW decomposition of the exterior **X** is nothing more than the non-abelian free resolution:

$$G \xleftarrow{\nu} F(a,b,c) \xleftarrow{\wedge} \mathfrak{F}_F(r_1,r_2,r_3) \xleftarrow{\wedge} \mathfrak{F}_F(I)$$

$$cb\bar{c}\bar{a} \xleftarrow{\wedge} r_1 \qquad r_1r_2r_3 \xleftarrow{\wedge} I$$

$$ac\bar{a}\bar{b} \xleftarrow{\wedge} r_2$$

$$ba\bar{b}\bar{c} \xleftarrow{\wedge} r_3$$

Please note that $\wedge^2 = 1$ and $\nu \wedge = 1$.

Wirtinger Generalized Presentation

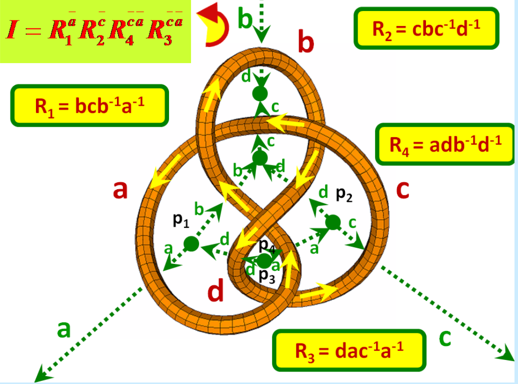
$$G \xleftarrow{\nu} F(a,b,c) \xleftarrow{\wedge} \mathfrak{F}_F(r_1,r_2,r_3) \xleftarrow{\wedge} \mathfrak{F}_F(I)$$

We denote the above non-abelian free resolution more cryptically as

$$(a,b,c : r_1,r_2,r_3 : I)$$

and call it a generalized presentation.

Wirtinger Generalized Presentation



$I = R_1^{\bar{a}} R_2^{\bar{c}} R_4^{\bar{c}\bar{a}} R_3^{\bar{c}\bar{a}}$

R_2 = cbc^{-1}d^{-1}

R_1 = bcb^{-1}a^{-1}

R_4 = adb^{-1}d^{-1}

R_3 = dac^{-1}a^{-1}

Wirtinger Generalized Presentation

$$(a,b,c,d : r_1,r_2,r_3,r_3 : I)$$

$$bc\bar{b}\bar{a} \xleftarrow{\wedge} r_1 \qquad r_1^a r_2^c r_4^{ca} r_3^{ca} \xleftarrow{\wedge} I$$

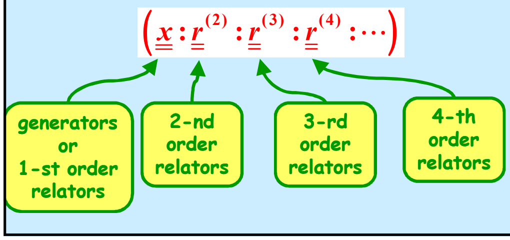
$$cb\bar{c}\bar{d} \xleftarrow{\wedge} r_2$$

$$da\bar{c}\bar{a} \xleftarrow{\wedge} r_3$$

$$ad\bar{b}\bar{d} \xleftarrow{\wedge} r_4$$

Terminology for Generalized Presentations

The current names **identities**, **identities of identities**, **identities of identities of identities**, etc. are too cumbersome. So we instead adopt the following terminology:



$$\left(\underline{\underline{x}} : \underline{\underline{r}}^{(2)} : \underline{\underline{r}}^{(3)} : \underline{\underline{r}}^{(4)} : \cdots\right)$$

generators or 1-st order relators

2-nd order relators

3-rd order relators

4-th order relators

Generalized Tietze Moves

There is only one generalized Tietze move for each order, namely:

n-th <u>order</u> <u>generalized</u> <u>Tietze</u> <u>move</u> T_n :

$$T_n : \left(\cdots : \underline{\underline{r}}^{(n)} : \underline{\underline{r}}^{(n+1)} : \cdots\right) \mapsto \left(\cdots : \underline{\underline{r}}^{(n)} \cup \sigma : \underline{\underline{r}}^{(n+1)} \cup \tau : \cdots\right)$$

where $\hat{\tau} = \sigma \xi^{-1}$ and $\widehat{\sigma} = \hat{\xi}$

and where $\xi \in \mathfrak{F}_F\left(\underline{\underline{r}}^{(n)}\right)$

Generalized Presentation Equivalence

The definition of generalized presentation equivalence is a straight forward exercise for the audience. So is the proof of the following theorem:

<u>Theorem</u>: Two generalized presentations are equivalent iff there is a finite sequence of generalized Tietze moves that transform one into the other.

The Geometry of Generalized Presentations

<u>Proposition</u>. Up to renaming and reordering, there exists a one-to-one correspondence between the set of finite generalized presentations and a set of monopointed CW-complexes.

$$\left(\underline{\underline{x}} : \underline{\underline{r}} : \underline{\underline{r}}^{(3)} : \underline{\underline{r}}^{(4)} : \cdots\right) \leftrightarrow K\left(\underline{\underline{x}} : \underline{\underline{r}} : \underline{\underline{r}}^{(3)} : \underline{\underline{r}}^{(4)} : \cdots\right)$$

The generalized tietze moves correspond to simple homotopy moves on the associated CW-complexes.

Moreover, two generalized presentations define CW complexes of the same simple homotopy type iff there is a finite sequence of Tieze moves which transforms one into the other.

The Geometry of the Fox Free Calculus

Let $\mathfrak{P} = \left(\underline{\underline{x}} : \underline{\underline{r}}^{(2)} : \underline{\underline{r}}^{(3)} : \cdots : \underline{\underline{r}}^{(n)}\right)$ be a generalized presentation, and let

$$K = K(\mathfrak{P})$$

be the corresponding CW-complex.

Let $G = \pi_1(K)$ the fundamental group of K.

Let $\nu : F(\underline{\underline{x}}) \to G$ be the epimorphism associated with $\left(\underline{\underline{x}} : \underline{\underline{r}}^{(2)}\right)$

Finally, let $\mathbb{Z}G$ be the group ring of G over the integers $\mathbb{Z}$.

The Geometry of the Fox Free Calculus

Let $\widetilde{K}$ be the universal cover of K , and let $\widetilde{\widetilde{K}} = \widetilde{K} \times Ker\,\nu$ be the non pathwise connected space above $\widetilde{K}$.

We now use the Fox free derivatives to construct a chain complex $C_* = C_*\left(\widetilde{\widetilde{K}}\right)$

Hence, $H_*\left(\widetilde{\widetilde{K}}\right) = H_* C_*$

Moreover, $H_*\left(\widetilde{K}\right) = H_*\left(\mathbb{Z}G \otimes_{\mathbb{Z}F} C_*\right)$

The Geometry of the Fox Free Calculus

The chain groups are defined as follows:

- The **0**-th chain group $C_0 = C_0(\infty)$ is defined as the free $\mathbb{Z}F$ -module generated by the **0**-cell ∞ .

- For **n > 0** , the **n**-th chain group $C_n = C_n\left(\underline{\underline{R}}^{(n)}\right)$ is defined as the free $\mathbb{Z}F$ -module generated by the set of **n**-cells $\underline{\underline{R}}^{(n)}$.

The Geometry of the Fox Free Calculus

The boundary morphisms are defined as follows:

- For **n = 0** ,

$$\begin{tikzcd} C_0(\infty) & C_1\left(\underline{\underline{X}}\right) \arrow[l, "\partial"'] \\ \left(x_j - 1\right)\infty & X_j \arrow[l] \end{tikzcd}$$

- For **n > 0** ,

$$\begin{tikzcd} C_{n-1}\left(\underline{\underline{R}}^{(n-1)}\right) & C_n\left(\underline{\underline{R}}^{(n)}\right) \arrow[l, "\partial"'] \\ \sum_k \left(\frac{\partial \hat{r}_j^{(n)}}{\partial r_k^{(n-1)}}\right) R_k^{(n-1)} & R_k^{(n)} \arrow[l] \end{tikzcd}$$

Where the Fox Free derivatives $\partial / \partial x_j$ are geometrically defined as follows:

Cell Decompositions of $K, \tilde{K}, \tilde{\tilde{K}}$

Epimorphism: $F\left(\underline{\underline{x}}\right) \xrightarrow{\nu} G = \pi_1 K$

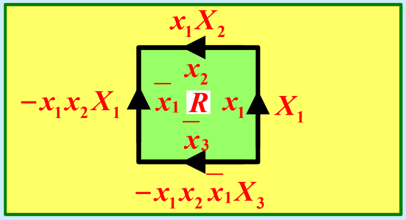
$$\begin{tikzcd}[column sep=small] \tilde{K} \times Ker\,\nu = & \tilde{\tilde{K}} \arrow[d] & \{w\infty\} \arrow[d] & \{wX_i\} \arrow[d] & \{wR_j\} \arrow[d] & \{wR_k^{(3)}\} \arrow[d] & \{wR_4^{(4)}\} \arrow[d] \\ \text{Univ. Cover} & \tilde{K} \arrow[d] & \{g\infty\} \arrow[d] & \{gX_i\} \arrow[d] & \{gR_j\} \arrow[d] & \{gR_k^{(3)}\} \arrow[d] & \{gR_4^{(4)}\} \arrow[d] \\ \text{CW-Complex} & K & \{\infty\} & \{X_i\} & \{R_j\} & \{R_k^{(3)}\} & \{R_\ell^{(4)}\} \\ & & \text{0-cells} & \text{1-cells} & \text{2-cells} & \text{3-cells} & \text{4-cells} \end{tikzcd}$$

$g \in G$

$w \in F\left(\underline{\underline{x}}\right)$

Recall $\mathfrak{P} = \left(\underline{\underline{x}} : \underline{\underline{r}}^{(2)} : \underline{\underline{r}}^{(3)} : \cdots : \underline{\underline{r}}^{(n)}\right)$ **and** $K = K(\mathfrak{P})$.

If, for example, $r \in \underline{\underline{r}}^{(2)}$ with $\hat{r} = x_1 x_2 \bar{x}_1 \bar{x}_3$, then the corresponding 2-cell R in $\tilde{\tilde{K}}$ is



$$\partial R = \left(1 - x_1 x_2\right) X_1 + x_1 X_2 - x_1 x_2 \bar{x}_1 X_3$$

$$= \frac{\partial \hat{r}}{\partial x_1} X_1 + \frac{\partial \hat{r}}{\partial x_2} X_2 + \frac{\partial \hat{r}}{\partial x_3} X_3$$

If, for example, $u \in \underline{\underline{r}}^{(3)}$ with $\hat{u} = r_1^{x_1} r_2^{x_3} \bar{r}_1^{x_1 x_3}$, then the boundary chain map of the corresponding 3-cell U in $\tilde{\tilde{K}}$ is:

$$\partial U = \left(x_1 - \hat{r}_1^{x_1} \hat{r}_2^{x_3} x_1 x_3 \hat{\bar{r}}\right) R_1 + \hat{r}_1^{x_1} x_3 R_2$$

$$= \left(\frac{\partial \hat{u}}{\partial r_1}\right)^{\wedge} R_1 + \left(\frac{\partial \hat{u}}{\partial r_2}\right)^{\wedge} R_2$$

For completeness, we give the standard algebraic definition of the Fox free $\partial / \partial x_j$

Algebraic Def. of Fox Free Derivative $\partial / \partial x_j$

Let G be a group, and let $\mathbb{Z}G$ denote the corresponding group ring over the ring of integers $\mathbb{Z}$.

Def. A derivative D in the group ring in the group ring $\mathbb{Z}G$ is defined as a map

$$D : \mathbb{Z}G \to \mathbb{Z}G$$

satisfying the following condition:

1) $D\left(\omega_1 + \omega_2\right) = D\omega_1 + D\omega_2$
2) $D\left(\omega_1 \omega_2\right) = \left(D\omega_1\right)\omega_2^o + \omega_1 D\omega_2$

where ${}^o : \mathbb{Z}G \to \mathbb{Z}G$ is the trivializer morphism which maps each element of G to 1 of $\mathbb{Z}$.

Algebraic Def. of Fox Free Derivative $\partial / \partial x_j$

Def.(Cont.) Let G be the free group $F\left(\underline{\underline{x}}\right)$. Then to each free generator $x_j \in \underline{\underline{x}}$, there corresponds a unique derivative

$$D_j = \partial / \partial x_j$$

in $\mathbb{Z}F\left(\underline{\underline{x}}\right)$, called the derivative with respect to x_j , which has the property

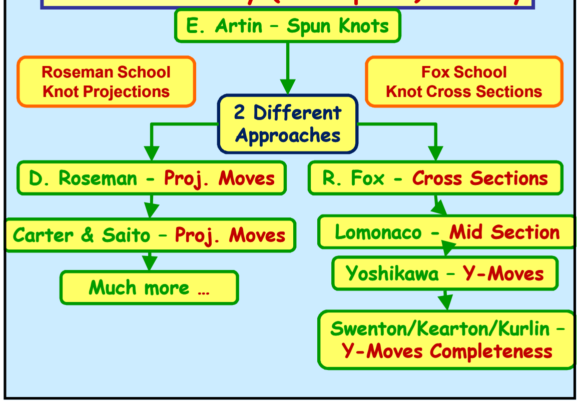
$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \quad \text{(Kronecker delta)}$$

2-Knots

Various Placement Problems

- 3-D Knot Theory
  $k : S^1 \to S^3$ 1-Knot $\left(S^3, kS^1\right)$
- 4-D Knot Theory
  $k : S^2 \to S^4$ 2-Knot $\left(S^4, kS^2\right)$
- 5-D Knot Theory
  $k : S^3 \to S^5$ 3-Knot $\left(S^5, kS^3\right)$

4-D Knot Theory (Incomplete) History



4-D Analog of the Apspericity of Knots ?

For 1-knots, asphericity implies the exterior X is an Eilenberg-MacLane space, i.e.,

$$X = K(\pi_1 X, 1)$$

Question. Want can be said about analog of Papa's ashphericity theorem for 2-knots?

4-D Analog of the Apspericity of Knots ?

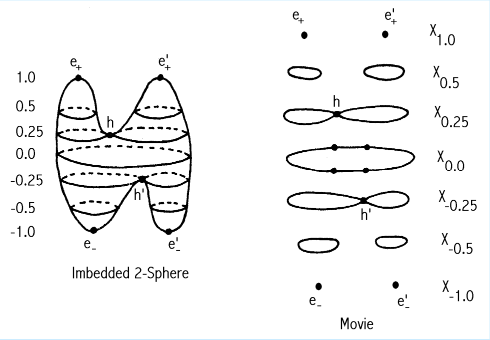
Def. A 2-knot (S^4, kS^2) is said to be quasi-aspherical (QA) if the third homology group of the universal cover of its exterior vanishes.

Theorem. (Lomonaco) If (S^4, kS^2) is QA, then the homotopy type of its exterior X is determined by its algebraic 3-type, i.e., by the triple consisting of:

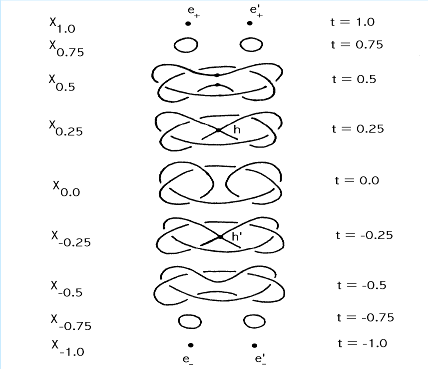
- $\pi_1 X$
- $\pi_2 X$ as a $\mathbb{Z}\pi_1 X$ -module
- The first k-invariant kX lying in $H^3(\pi_1 X; \pi_2 X)$

The Cross sectional Approach to 2-Knots

Edwin Abbott's Flatland

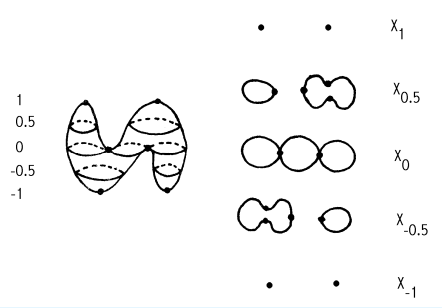


3-D Land

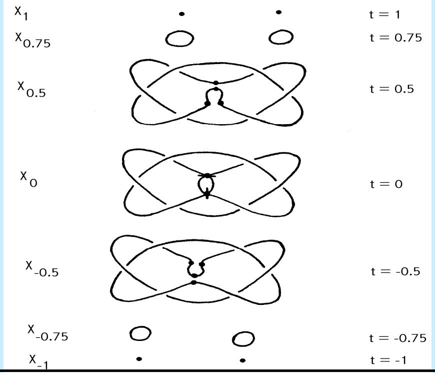


The Midsection Representation 2-Knots

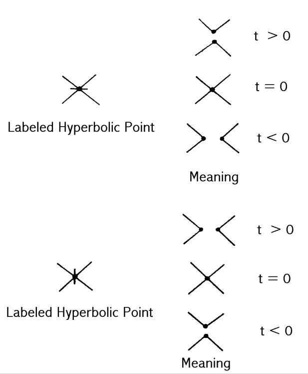
Edwin Abbott's Flatland



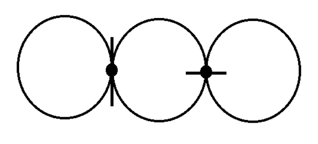
3-D Land



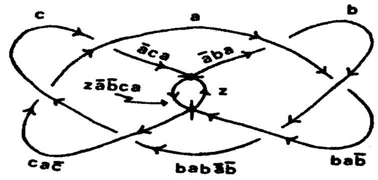
Midsection Labeling Scheme



Midsection Representation of a 2-sphere

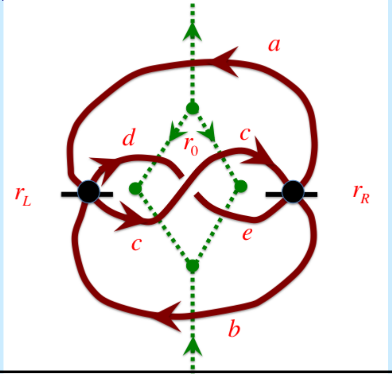


Midsection Rep of a Spun Trefoil



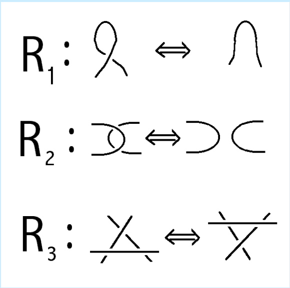
$$\begin{cases} \pi_1 K = (\ a, b : abab^{-1}a^{-1}b^{-1} = 1\ ) \\ \pi_2 K = (\ \partial U_{\square} :\ (1\text{-}a+ba)\,\partial U_{\square} = 0\ ) \\ H^3(\ \pi_1 K ; \pi_2 K\ ) = 0 \quad \text{and} \quad \underline{\underline{k}}K = 0 \end{cases}$$

Midsection of $\left(S^4, k\mathbb{R}P^2\right)$

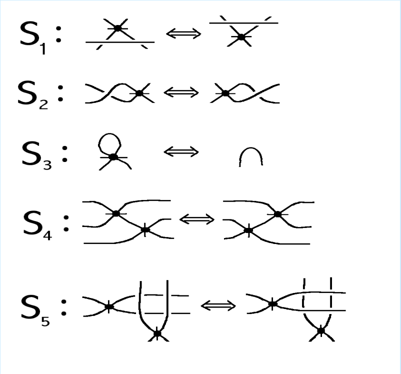


Midsection Moves on 2-Knots

Reidemeister Moves



Yoshikawa Moves



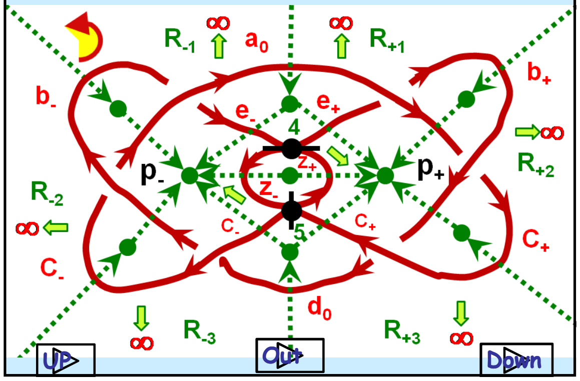
The Geometry of Group Presentations

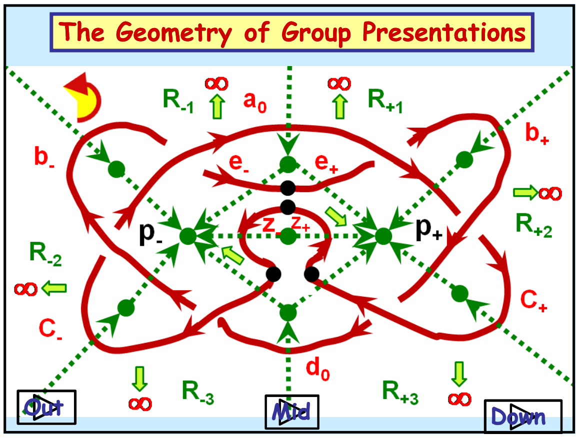
Theorem(Yoshikawa). The Reidemeister and Yoshikawa moves on the midsection representation of a 2-knot preserve knot type.

Theorem(Swenton/Keartin/Kurlin). Two 2-knot midsections represent the same 2-knot iff there is a finite sequence of Reidemeister and Yoshikawa moves which transforms one into the other.

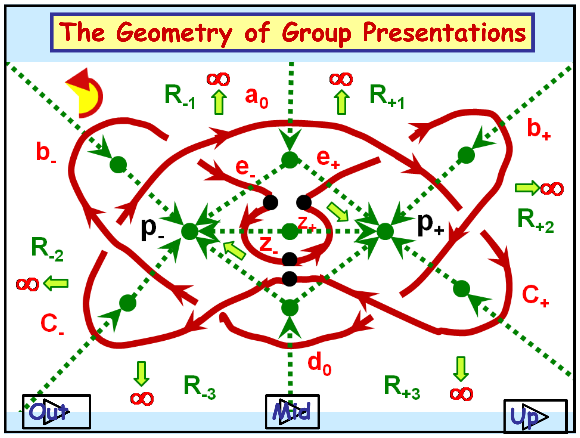
The Wirtinger Generalized Presentation of 2-Knots ???

The Geometry of Group Presentations

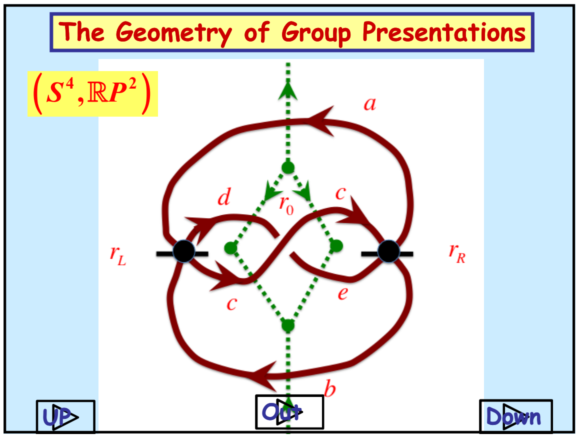




The Geometry of Group Presentations

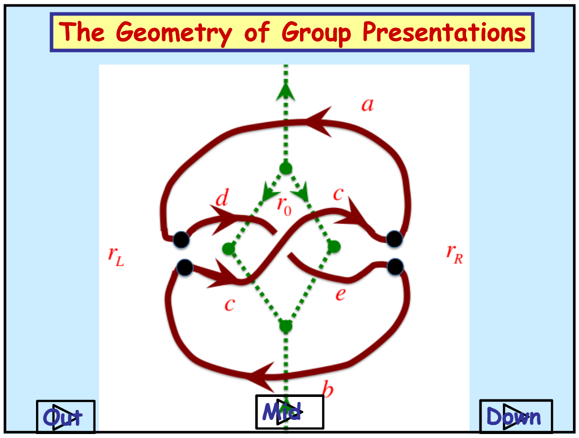


The Geometry of Group Presentations

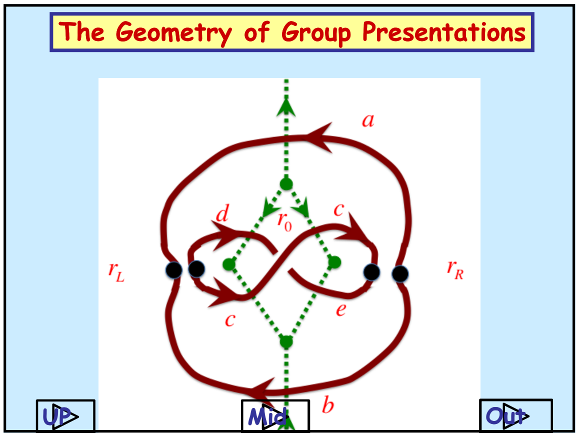


The Geometry of Group Presentations

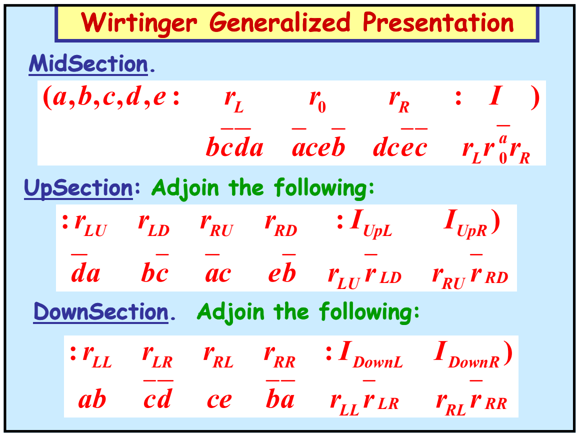
$\left(S^4, \mathbb{R}P^2\right)$



The Geometry of Group Presentations



The Geometry of Group Presentations



Wirtinger Generalized Presentation

MidSection.

$$\left(a,b,c,d,e : \begin{array}{ccc} r_L & r_0 & r_R \\ \bar{b}\bar{c}da & \bar{a}ce\bar{b} & \bar{d}\bar{c}ec \end{array} : \begin{array}{c} I \\ r_L\bar{r}_0^{a}r_R \end{array}\right)$$

UpSection: Adjoin the following:

$$: \begin{array}{cccc} r_{LU} & r_{LD} & r_{RU} & r_{RD} \\ \bar{d}a & b\bar{c} & \bar{a}c & e\bar{b} \end{array} : \begin{array}{cc} I_{UpL} & I_{UpR} \\ r_{LU}\bar{r}_{LD} & r_{RU}\bar{r}_{RD} \end{array} \Big)$$

DownSection. Adjoin the following:

$$: \begin{array}{cccc} r_{LL} & r_{LR} & r_{RL} & r_{RR} \\ ab & \bar{c}\bar{d} & ce & \bar{b}\bar{a} \end{array} : \begin{array}{cc} I_{DownL} & I_{DownR} \\ r_{LL}\bar{r}_{LR} & r_{RL}\bar{r}_{RR} \end{array} \Big)$$

Talk Overview

This talk is a description of a research program which is best summarized by the following two questions:

Open Question 1

How does one piece together the Wirtinger generalized presentations of all the exterior cross sections of a 2-knot $\left(S^4, kS^2\right)$ into a generalized presentation of the exterior of the entire 2-knot ???

Open Question 2

Given a generalized presentation of the exterior X of a 2-knot $\left(S^4, kS^2\right)$, how does one read off from this generalized presentation the second homotopy group $\pi_2 X$ (as $\mathbb{Z}\pi_1 X$ -module), and the k-invariant $kx \in H^3\left(\pi_1 X; \pi_2 X\right)$???

Conclusion ???

The answers are …

Upcoming
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Guest Editors

Kauffman & Lomonaco

This talk based on:

Lomonaco, Samuel J., Jr., Five dimensional knot theory, in "Low Dimensional Topology, AMS CONM/20, Providence, Rhode Island, (1984), pp 249 - 270

Lomonaco, Samuel J., Jr., The homotopy groups of knots I. How to compute the algebraic 3-type, Pacific Journal of Mathematics, Vol. 95, No. 2 (1981), pp 349 - 390

Lomonaco, Samuel J., Jr., Homology of group systems with applications to low dimensional topology, Bulletin of the American Mathematics Society, Vol. 13, No. 3 (1980), pp 1049 - 1052.

Lomonaco, S.J., Jr., The second homotopy group of a spun knot, Topology, Vol. 8 (1969), pp 95 - 98

And also based on:

Fox, R.H., A quick trip through knot theory, in "Topology of 3-Manifolds and Related Topics," ed. by M.K. Fort, Jr., Prentice-Hall, Englewoods Cliffs, New Jersey, (1962), 120-167.

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Lomonaco, Samuel J., Jr., The fundamental ideal and Pi2 of higher dimensional knots, AMS Proc., 88, (1973), 431 - 433.
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Roseman, Dennis, Reidemeister type moves for surfaces in four space, preprint
Roseman, Dennis, Projections of knots, Fund. Math. 89 , no. 2, (1975), pp. 99-110.