KNOTS, LINKS and GEOMETRY

Herman Gluck

Knots in Washington, May 2013

My mathematical life began with knots and low-dim'l topology, then moved into differential geometry, and later came back to knots and links .... but viewed through a geometer's eyes, and with application to molecular biology, fluid dynamics and plasma physics.

The Crab Nebula

KNOTS

1. Knot theory in the presence of curvature

LINKS WITH TWO COMPONENTS

2. The Gauss map and Gauss linking integral

3. Writhing of knots and helicity of vector fields

LINKS WITH THREE COMPONENTS

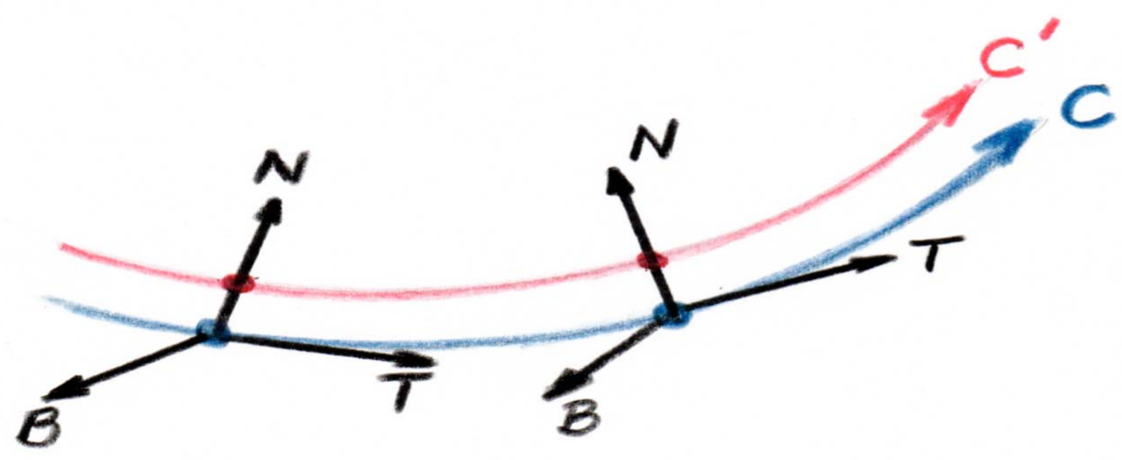
4. Generalized Gauss maps and integrals for three-component links

KNOTS

1. Knot theory in the presence of curvature.

In the mid 1990s, my former student Liu-Hua Pan, then an assistant professor at Indiana, created the subject of "knot theory in the presence of curvature".

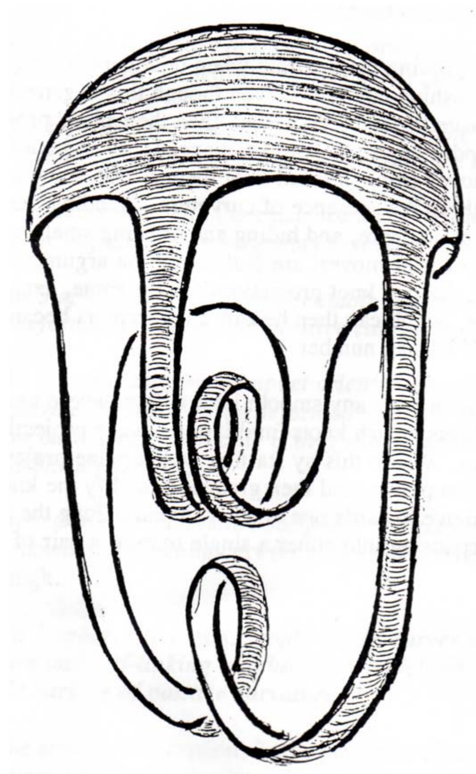
In this theory, knots have nowhere vanishing curvature.



Curve C with Frenet frame and push-off C'

The ***self-linking number*** of a knot with nowhere-vanishing curvature is the linking number with its push-off in the direction of its principal normal vector.

The surfaces in this theory have positive curvature.



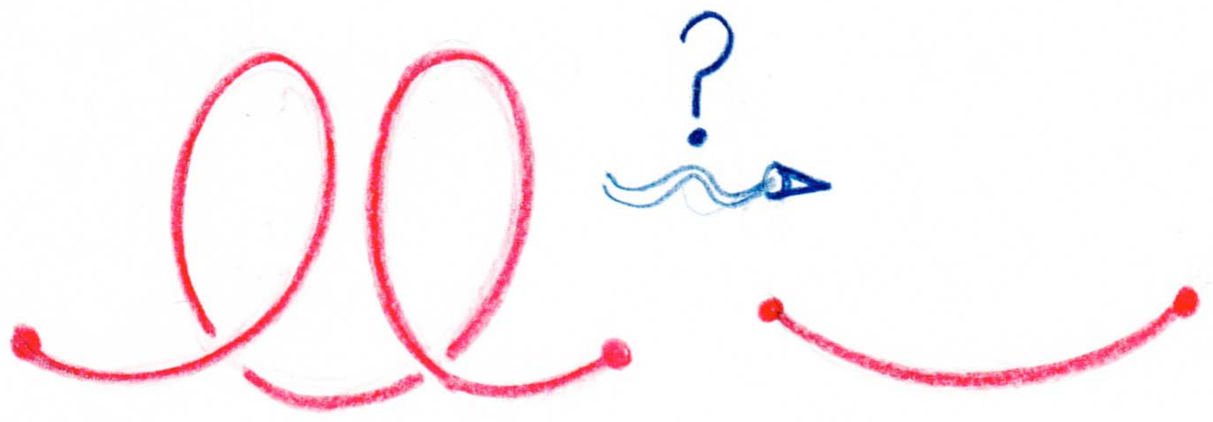
Curves on them have nowhere vanishing curvature.

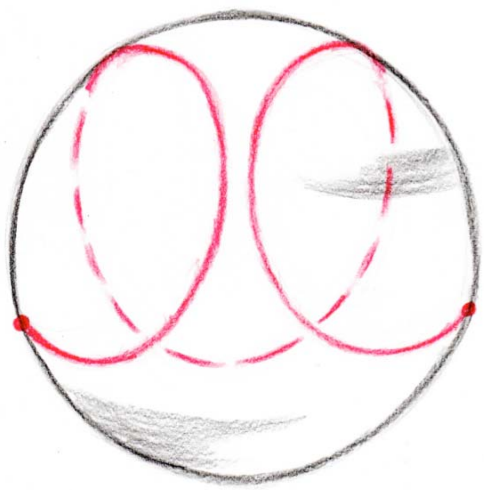
THREE THEOREMS (with Liu-Hua Pan).

Theorem 1. Any two smooth simple closed curves in 3-space, each having nowhere vanishing curvature, can be deformed into one another through a one-parameter family of such curves if and only if they have the same knot type and the same self-linking number.

This result was known to Bill Pohl, and also follows from the parametric h-principle of Eliashberg and Gromov.

Test of Theorem 1.





Theorem 2. In 3-space, any compact orientable surface with non-empty boundary can be deformed into one with positive curvature.

Any two such surfaces with positive curvature can be deformed into one another through surfaces of positive curvature if and only if they can be deformed into one another through ordinary surfaces, preserving their natural orientations.

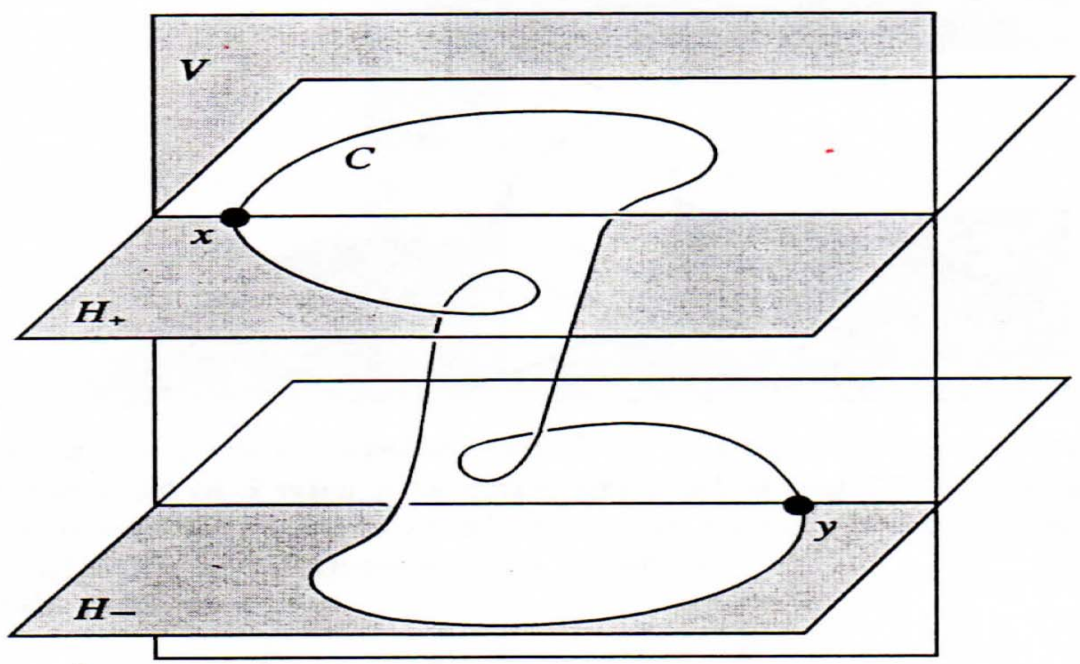
Problem 26 (Yau's 1993 Open Problems in Geometry).

What is the condition for a space curve to be the boundary of a surface of positive curvature?

A necessary condition is that the curve have nowhere vanishing curvature and self-linking number zero.

Is that enough?

Theorem 3. In 3-space, there exist simple closed curves with nowhere vanishing curvature and self-linking number zero, which do not bound any compact orientable surface of positive curvature.



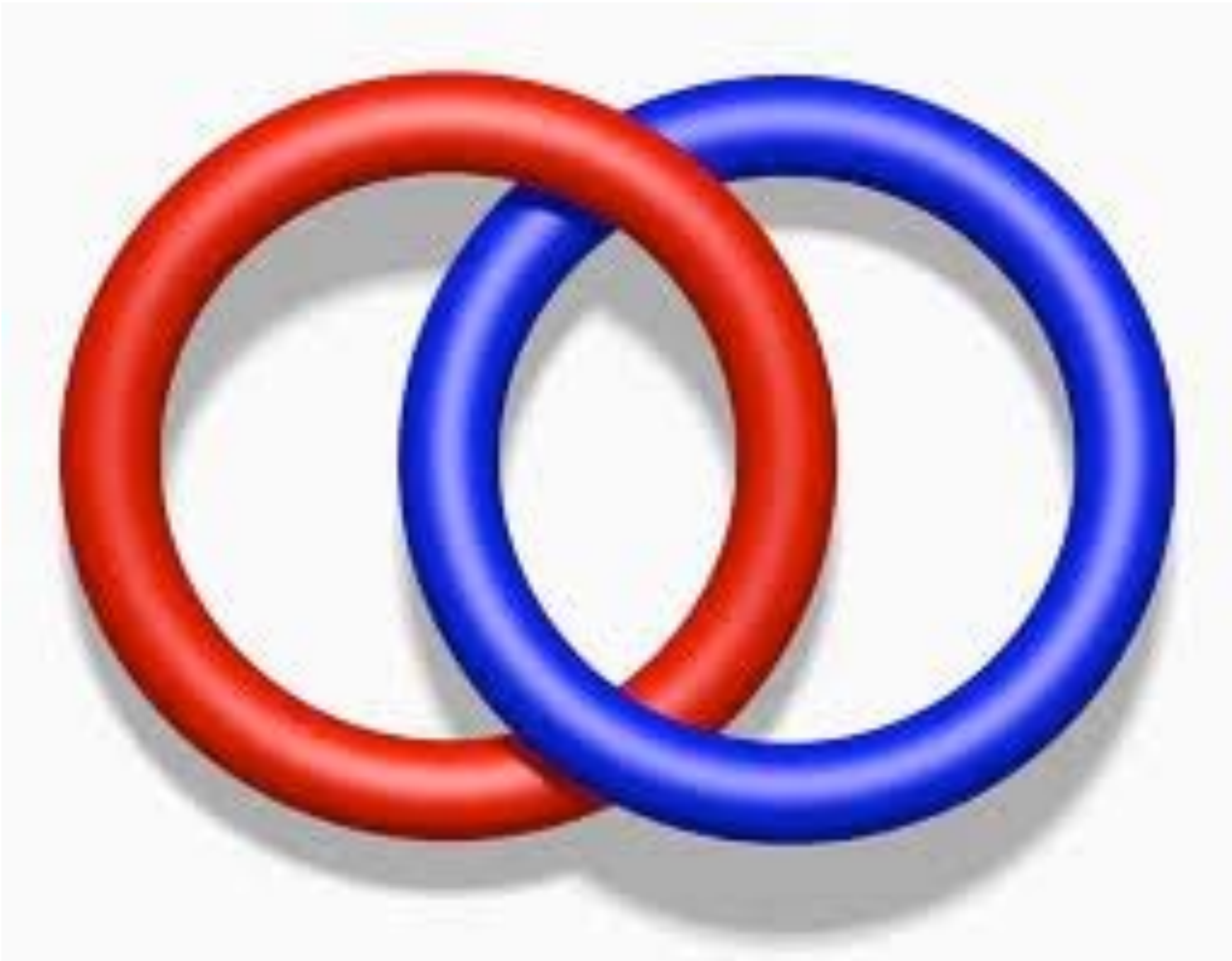
WHERE TO LEARN MORE.

Embedding and knotting of positive curvature surfaces in 3-space (with Liu-Hua Pan), Topology 37 (1998) 851-873.

Embedding and knotting of flat compact surfaces in 3-space (Peter Rogen), Comment. Math. Helv. 76 (2001), 589-606.

Strictly convex submanifolds and hypersurfaces of positive curvature (Mohammad Ghomi), J. Diff. Geom., 57 (2001), 239-271.

LINKS WITH TWO COMPONENTS



2. The Gauss map and Gauss linking integral.

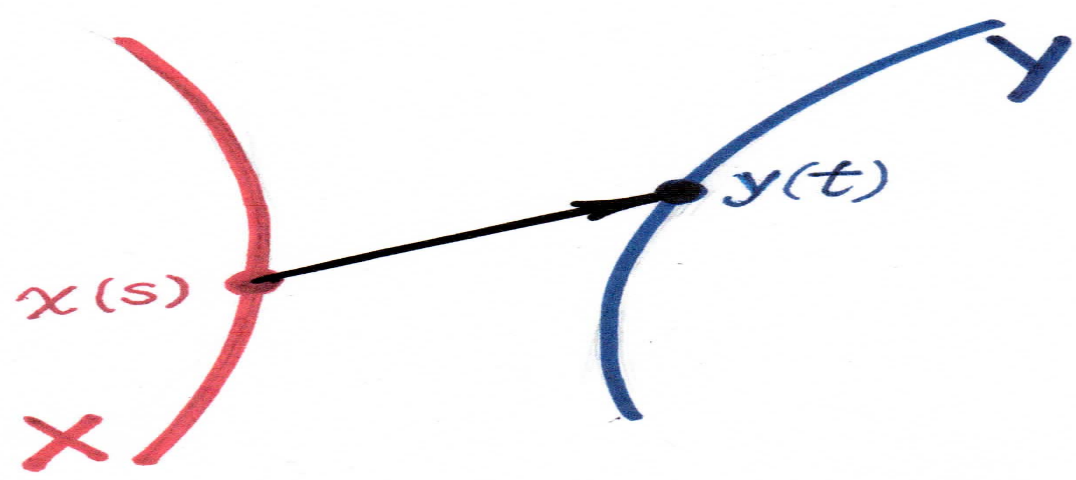
Carl Friedrich Gauss (1777-1855)

The Gauss map. Given a link L with two components,

$$X = \{x(s) : s \in S^1\} \quad \text{and} \quad Y = \{y(t) : t \in S^1\},$$

its Gauss map $g_L : S^1 \times S^1 \to S^2$ is defined by

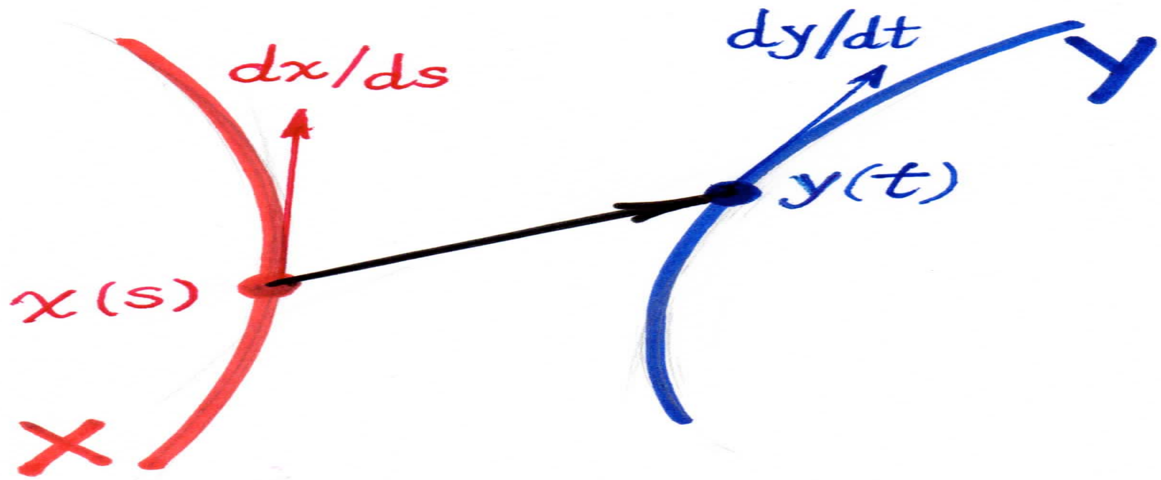
$$g_L(s, t) = \frac{y(t) - x(s)}{|y(t) - x(s)|}$$



The linking number $Lk(X, Y)$ = degree of g_L .

The Gauss linking integral.

$$\mathrm{Lk}(X, Y) = 1/4\pi \int_{S^1 \times S^1} dx/ds \times dy/dt \bullet \frac{x(s) - y(t)}{|x(s) - y(t)|^3}\, ds\, dt$$



Naturality of the Gauss map and integral.

(1) The Gauss map $\mathbf{g}_{L} : \mathbf{S}^1 \times \mathbf{S}^1 \rightarrow \mathbf{S}^2$ is equivariant w.r.t. orientation-preserving rigid motions h of R^3 :

$$\mathbf{g}_{\mathbf{h}(L)} = \mathbf{h} \circ \mathbf{g}_{L} .$$

(2) The Gauss integrand $\mathbf{dx/ds \times dy/dt \bullet (x-y)/|x-y|^3}$ is invariant under orientation-preserving rigid motions h of R^3 , i.e, it is the same for the link *L* as it is for h(*L*) .

(3) These features are crucial for application to physics.

3. Linking, Writhing, Helicity and Magnetic Fields.

$$Lk(X, Y) = 1/4\pi \int_{X\times Y} dx/ds \times dy/dt \bullet \frac{x(s) - y(t)}{|x(s) - y(t)|^3} \, ds\, dt$$

$$Wr(X) = 1/4\pi \int_{X\times X} dx/ds \times dy/dt \bullet \frac{x(s) - y(t)}{|x(s) - y(t)|^3} \, ds\, dt$$

$$Hel(V) = 1/4\pi \int_{\Omega\times\Omega} V(x) \times V(y) \bullet \frac{x - y}{|x - y|^3} \, dx\, dy$$

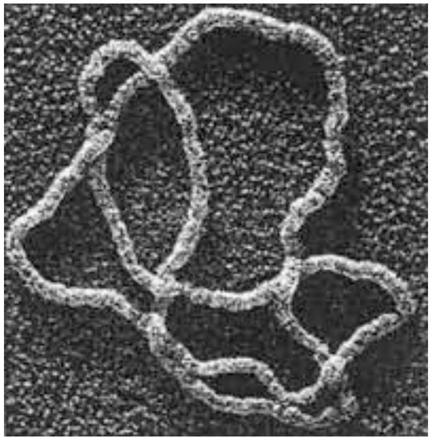
$$BS(V)(y) = 1/4\pi \int_{\Omega} V(x) \times (y - x) / |y - x|^3 \, dx$$

The ***writing number***

$$Wr(X) \;=\; 1/4\pi \int_{X\times X} dx/ds \times dy/dt \bullet (x - y)/|x - y|^3 \, ds\, dt$$

of a knot X was introduced by Calugareanu (1959-61) and named by Fuller (1971), and measures the extent to which the knot wraps and coils around itself.

It is important in the study of DNA and of the enzymes which affect it.



Gheorghe Calugareanu (1902 - 1976)

The ***helicity***

$$\mathbf{Hel(V) = 1/4\pi \int_{\Omega\times\Omega} V(x) \times V(y) \bullet (x - y)/|x - y|^3\, dx\, dy}$$

of a vector field was introduced by Woltjer in 1958 and named by Moffatt in 1969, and measures the extent to which its orbits wrap and coil around one another.

Arnold showed in 1974 that the helicity of V is the average linking number between its various orbits.

Helicity is related to magnetic fields via the formula

$$\mathbf{Hel(V) = \int_{\Omega} V \bullet BS(V)\, d(vol)}$$

Lodewijk Woltjer

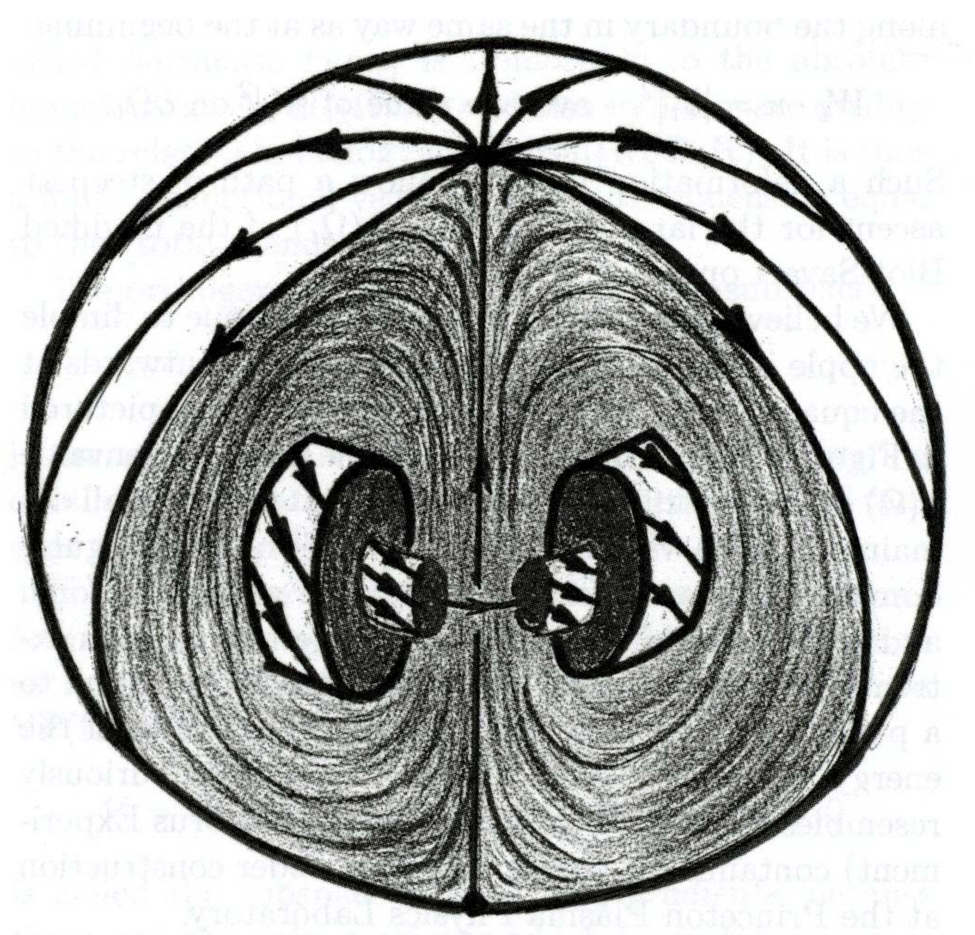
Henry Keith Moffatt

How did Woltjer use helicity?

Woltjer showed that the helicity of a magnetic field remains constant under ideal magnetohydrodynamic evolution ...

... and that when nonzero it provides a lower bound for the energy of the field under this evolution.

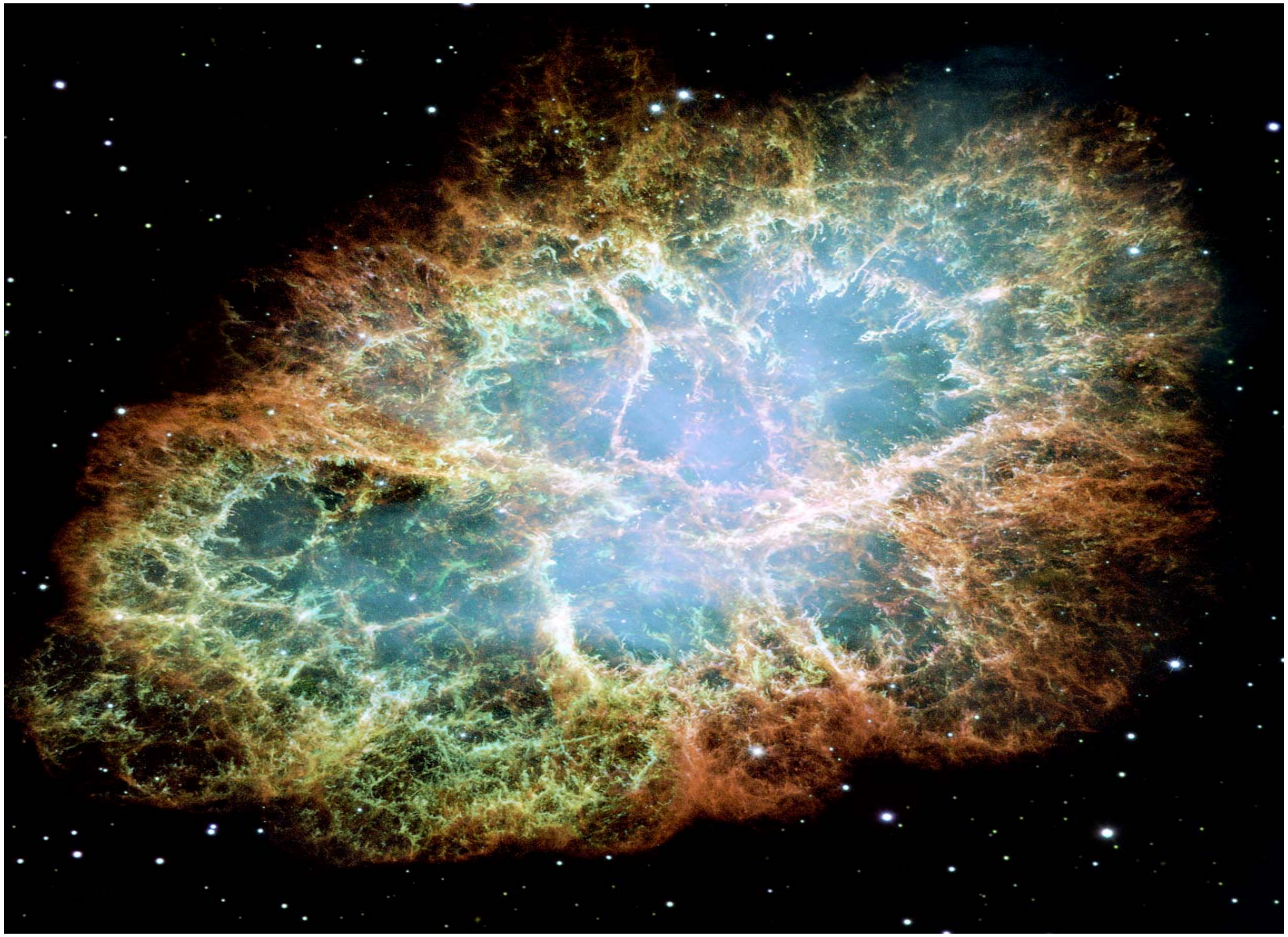
He used this to compute a model for the magnetic field in the Crab Nebula, minimizing energy for given helicity.



Woltjer's model for the magnetic field in the Crab Nebula

divergence-free, tangent to boundary,

with minimum energy for given nonzero helicity



The Crab Nebula

The Crab Nebula, mostly in the X-ray spectrum

WHERE TO LEARN MORE.

The asymptotic Hopf invariant and its applications
(V. I. Arnold), English translation in Selecta Math. Sov. 5(4) (1986), 327-342. Original in Russian, Erevan (1974).

Influence of Geometry and Topology on Helicity
(with Jason Cantarella, Dennis DeTurck, Mikhail Teytel) Magnetic Helicity in Space and Laboratory Plasmas, American Geophysical Union (1999).

Isoperimetric problems for the helicity of vector fields and the Biot-Savart and curl operators
(with Jason Cantarella, Dennis DeTurck, Mikhail Teytel) Journal of Math. Physics 41 (2000), 5615-5641.

LINKS WITH THREE COMPONENTS

The Borromean rings

4. Generalized Gauss maps and integrals for three-component links.

The helicity of a vector field provides, via eigenvalue estimates for the Biot-Savart operator BS, a useful lower bound for its energy when the helicity is nonzero.

When the helicity is zero, it is sometimes possible to evolve the field so that its energy decreases towards zero.

The dream of Arnold and Khesin.

"The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order $\leq k$ have zero value for a given field and there exists a nonzero invariant of order $k+1$, this nonzero invariant provides a lower bound for the field energy."

Since the helicity of a vector field is related to the ordinary linking of two curves, the next-higher-order helicity is expected to be related to the ***triple linking of three curves***.

Vladimir Igorevich Arnold (1937 - 2010)

Boris Khesin

Milnor's classification of three-component links up to link homotopy.

Three-component links in the 3-sphere S^3 were classified up to ***link homotopy*** - *a deformation during which each component may cross itself but distinct components remain disjoint* - by John Milnor in his senior thesis (publ. 1954).

A complete set of invariants is given by the pairwise linking numbers p , q and r of the components, and by the ***triple linking number*** μ , which is a residue class mod gcd(p, q, r).

The generalized Gauss map for a three-component link in S^3 .

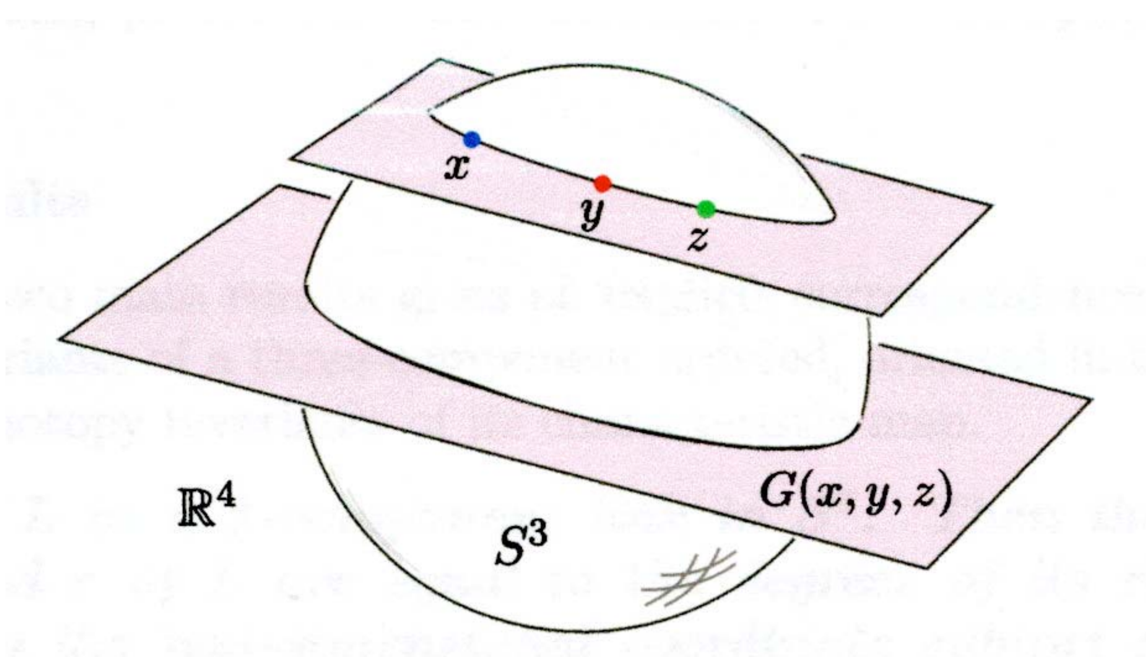
Given a three-component link L in S^3 , we will define a generalized Gauss map

$$\mathbf{g}_L : S^1 \times S^1 \times S^1 \rightarrow S^2$$

which is a natural successor to the classical Gauss map for a two-component link in R^3 .

Let x , y and z be three distinct points on $S^3 \subset R^4$.

They cannot lie on a straight line in R^4 , so must span a 2-plane there.



Orient this plane so that the vectors $x - z$ and $y - z$ form a positive basis, and then translate the plane so that it passes through the origin.

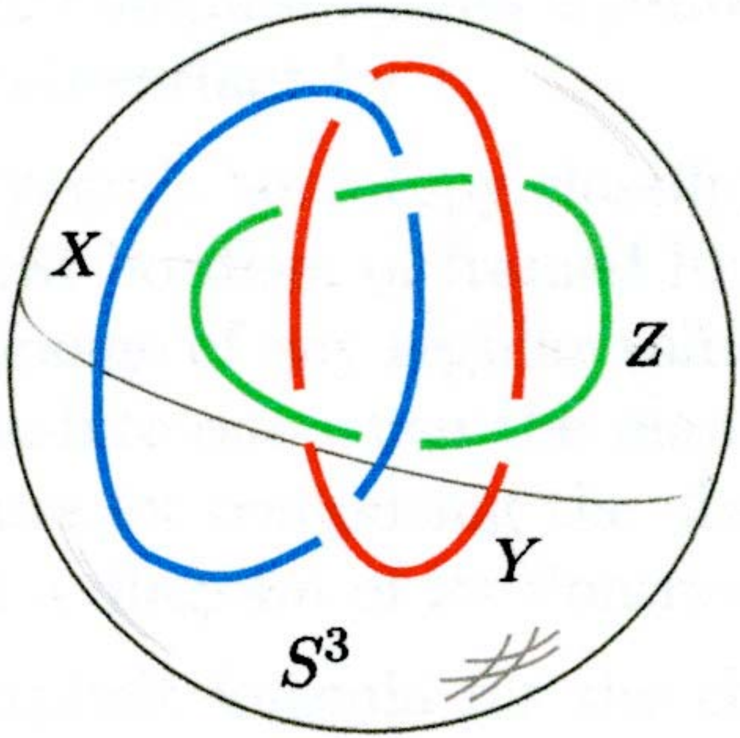
The result is an element $G(x, y, z)$ of the Grassmann manifold

$$G_2R^4 = SO(4) / \big(SO(2) \times SO(2)\big) \cong S^2 \times S^2$$

of all oriented 2-planes through the origin in R^4.

Now let L be a link in S^3 with components

$X = \{x(s): s \in S^1\}$, $Y = \{y(t): t \in S^1\}$, $Z = \{z(u): u \in S^1\}$.



The ***generalized Gauss map***

$$\mathbf{g}_L : \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \rightarrow \mathbf{S}^2$$

is then defined by

$$\mathbf{g}_L(\mathbf{s}, \mathbf{t}, \mathbf{u}) = \pi\, \mathbf{G}\big(\mathbf{x}(\mathbf{s}), \mathbf{y}(\mathbf{t}), \mathbf{z}(\mathbf{u})\big)\ ,$$

where $\pi\colon G_2R^4 \cong S^2 \times S^2 \rightarrow S^2$ is one of the projections.

Key properties of the generalized Gauss map.

(1) The map g_L is $SO(4)$-equivariant.

(2) Link homotopies of L become homotopies of g_L .

(3) g_L is "sign symmetric" under permutations of the components of L .

We will see that g_L is a natural generalization of the classical Gauss map $S^1 \times S^1$ for two-component links in R^3 .

Pontryagin's classification of maps of $S^1 \times S^1 \times S^1 \to S^2$ up to homotopy.

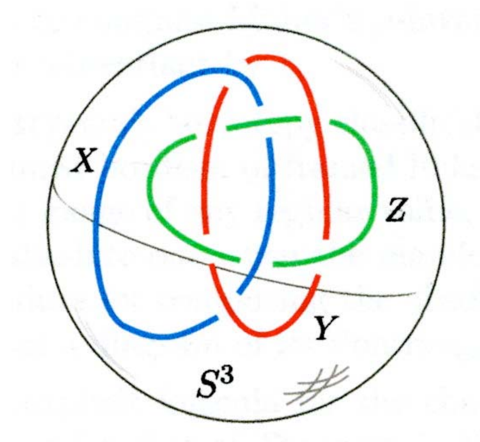
Lev Semenovich Pontryagin (1908 - 1988)

Classifying maps $S^1 \times S^1 \times S^1 \rightarrow S^2$ up to homotopy.

A complete set of invariants is given by the degrees p , q , r of the restrictions to the 2-dim'l coordinate subtori, and by the residue class ν of one further integer mod $2\,\mathrm{gcd}(p, q, r)$, the ***Pontryagin invariant*** of the map, which is an analogue of the Hopf invariant for maps from $S^3 \rightarrow S^2$.

Our Theorem A.

Let L be a three-component link in S^3 .



Then the pairwise linking numbers p, q and r of L are equal to the degrees of its characteristic map $g_L: S^1 \times S^1 \times S^1 \to S^2$ on the two-dim'l coord subtori, while twice Milnor's μ-invariant for L is equal to Pontryagin's ν-invariant for g_L mod 2 gcd(p, q, r) .

Credit where credit is due.

All these triple linking theorems are joint work with

Dennis DeTurck (Penn)

Rafal Komendarczyk (Tulane)

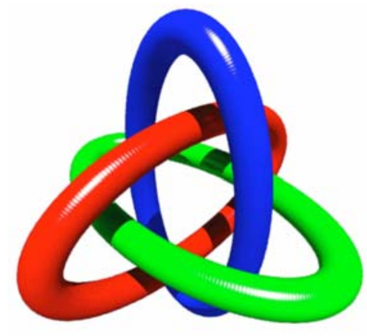
Paul Melvin (Bryn Mawr)

Clayton Shonkwiler (U Georgia)

Shea Vela-Vick (LSU)

Our Theorem B.

Let L be a three-component link in S^3, whose pairwise linking numbers p , q and r are all zero.



Then Milnor's μ-invariant of L is given by

$$\mu(L) = 1/2 \int_{T^3} d^{-1}(\omega_L) \wedge \omega_L ,$$

where $d^{-1}(\omega_L)$ is any 1-form on T^3 whose exterior derivative is ω_L .

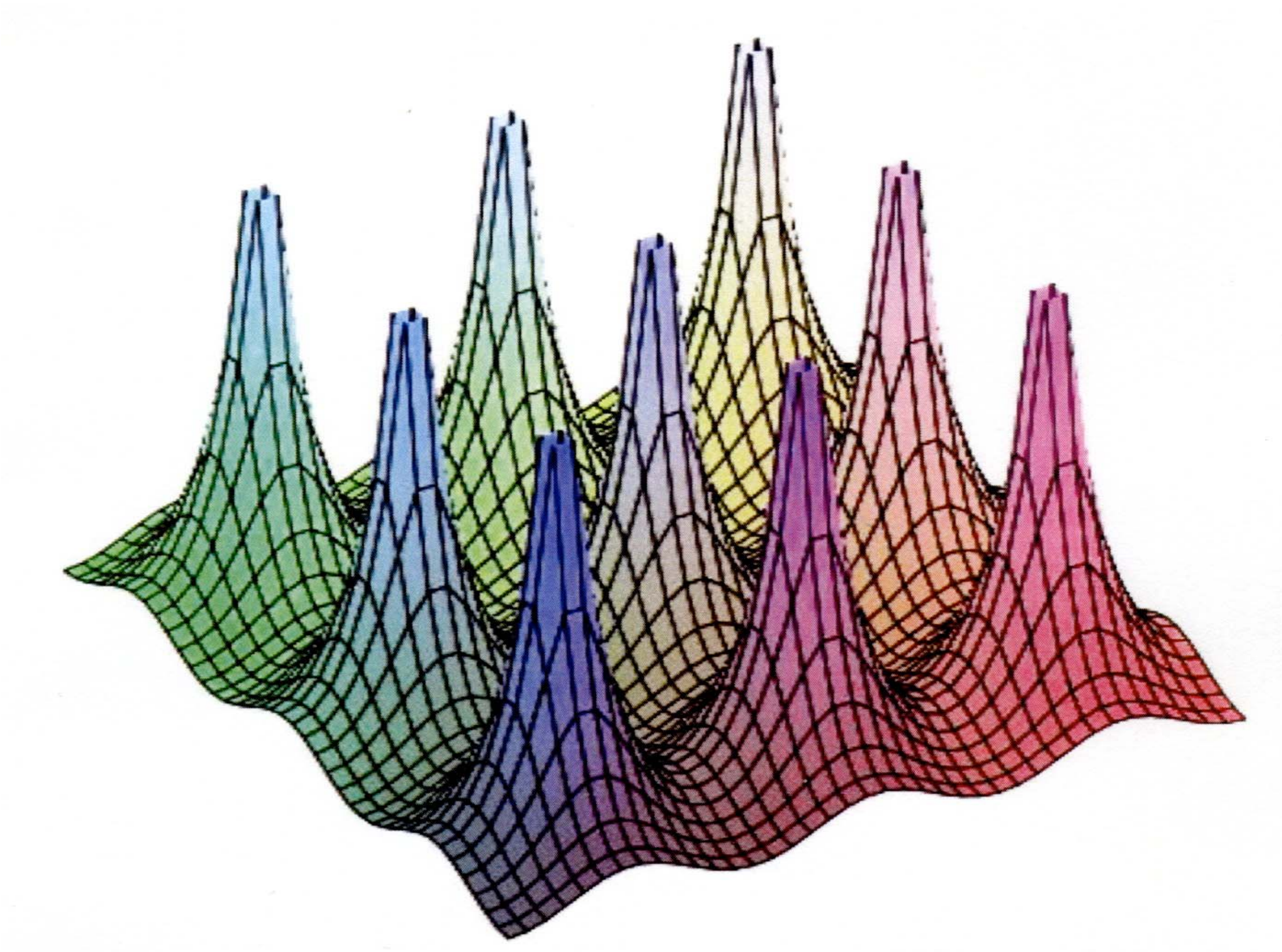
Notation in Theorem B.

ω = normalized Euclidean area 2-form on S^2

$\omega_L = g_L^*\omega$ is its pullback to the 3-torus T^3 under the generalized Gauss map $g_L : T^3 \to S^2$

The closed 2-form ω_L on T^3 is exact, because its periods are p, q and r, and they are all zero.

An explicit formula for the "anti-derivative" $d^{-1}(\omega_L)$ of least L^2 norm is given in terms of the fundamental solution of the scalar Laplacian on T^3.



Fundamental soln of scalar Laplacian on 2-torus

Our Theorem A in R^3 instead of S^3 .

To transplant all of this so that it deals with three-component links L in R^3 , we need a corresponding generalized Gauss map

$$\mathbf{g_L : S^1 \times S^1 \times S^1 \rightarrow S^2}$$

equivariant with respect to the group $\mathrm{Isom}^+(R^3)$ of orientation-preserving rigid motions of R^3 .

This map was found last year by **Haggai Nuchi**, a grad student at Penn.

WHERE TO LEARN MORE.

Link groups (John Milnor),
Annals of Math. 59 (1954). 177 - 195

A classification of mappings of the three-dim'l complex into the two-dimensional sphere
(Lev Pontryagin) Mat. Sbornik 9(51) (1941), 331 - 363.

Generalized Gauss maps and integrals for three-component links: Toward higher helicities for magnetic fields and fluid flows, Part 1
(with Dennis DeTurck, Rafal Komendarczyk, Paul Melvin, Clayton Shonkwiler and David Shea Vela-Vick),
to appear in Journal of Math. Physics (2013)

Generalized Gauss maps ... fluid flows, Part 2
(same authors + Haggai Nuchi),
to appear in Algebraic and Geometric Topology (2013).

The third order helicity of magnetic fields
via link maps (Rafal Komendarczyk),
Commun. Math. Physics 292 (2009), 431 - 456.

The third order helicity ... via link maps, Part II
(Rafal Komendarczyk), J. Math Physics 51 (2010).

I WILL BE HERE FOR THE NEXT THREE DAYS,

SO IF YOU WANT TO CHAT ABOUT ANY OF THIS,

I'LL LOOK FORWARD TO THAT.