Sequential change point detection in linear quantile regression models

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\textbf{A B S T R A C T}

We develop a method for sequential detection of structural changes in linear quantile regression models. We establish the asymptotic properties of the proposed test statistic, and demonstrate the advantages of the proposed method over existing tests through simulation.

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\section{1. Introduction}

For applications where the relationship between the response and covariates has a structural change at a certain point, the change may occur at the tails of the response distribution but not at the center. The conventional mean regression method for change point detection cannot be used to identify such structural changes at tails or may be lack of power in distributions with heavy tails. To provide a more robust testing procedure and obtain a more comprehensive view of structural changes, we focus on linear quantile regression models.

Several researchers have studied change point detection and estimation for quantile regression models, for instance, Bai (1996), Su and Xiao (2008), Qu (2008), Oka and Qu (2011), Furno (2007) and Furno (2012), to name a few. These work for quantile regression all focused on detecting changes in observations within a fixed length in a retrospective way. These retrospective quantile methods cannot be applied to the sequential data where new data arrive steadily, because the replication of such tests yields a procedure that rejects a true null hypothesis of no change with probability approaching one; see Robbins (1970). To our knowledge there exists little work for sequential change point detection in quantile regression models. Among some related work, Kouklová (2008) proposed a $L_1$-based monitoring procedure in linear regression models, and Chochola et al. (2013) discussed change point monitoring based on M-estimators. We develop a new procedure for sequentially monitoring structural changes in linear quantile regression models.

Let $y$ be the response variable and $x$ be the $p$-dimensional covariate vector with the first element 1. Denote $Q_y(\tau | x) = \inf\{y : F_y(y | x) \geq \tau\}$ as the $\tau$th conditional quantile of $y$ given $x$, where $F_y(\cdot | x)$ is the conditional distribution of $y$ given $x$.

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Let \( \{y_i, x_i, i \geq 1\} \) denote a sample with \( i \) representing a time index or some other ordering. We consider the linear quantile regression model

\[
Q_\tau(x) = x^T \beta_{i, \tau}, \quad i \geq 1,
\]

(1)

where \( \beta_{i, \tau} \) is the \( p \)-dimensional unknown quantile coefficient vector. We are interested in monitoring the changes of the effects of \( x \) on the quantiles of \( y \) over time, that is, monitoring the consistency of \( \beta_{i, \tau} \) over \( i \).

In Section 2, we present the proposed methods for monitoring change points at a single quantile level or across quantiles. The performance of the proposed methods is assessed through a simulation study in Section 3. The technical proofs are provided in the Supplementary Material (see Appendix A).

2. Proposed method

2.1. Sequential changepoint detection at a single quantile

We assume that there exists a historical data of size \( m \) such that \( \beta_{1, \tau} = \cdots = \beta_{m, \tau} = \beta_0^\tau \). This assumption was called the “noncontamination” assumption in Chu et al. (1996). The historical data is used for obtaining an estimate for the pre-change regression coefficient \( \beta_0^\tau \). At a given quantile level \( \tau \in (0, 1) \), we are interested in monitoring the future incoming observations sequentially for a change in the regression coefficient, that is, testing the null hypothesis

\[
H_0: \beta_{i, \tau} = \beta_0^\tau, \quad \text{for } i \geq m + 1,
\]

against the alternative hypothesis

\[
H_1: \beta_{i, \tau} = \begin{cases} 
\beta_i^\tau, & \text{for } m + 1 \leq i < m + k^* \\
\beta_0^\tau, & \text{for } i \geq m + k^*,
\end{cases}
\]

where \( k^* \geq 1 \) is the unknown change point, and \( \beta_0^\tau \neq \beta_i^\tau \) are the unknown pre- and post-change coefficients.

Let \( \hat{\beta}_i \) be the quantile coefficient estimator of \( \beta_i^\tau \) based on the historical data, that is, \( \hat{\beta}_i = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{m} \rho_\tau(y_i - x_i^T \beta) \), where \( \rho_\tau(u) = u(\tau - 1(u < 0)) \) is the quantile loss function; see Koenker and Bassett (1978). The building block of our monitoring process is the following subgradient-based CUSUM-type process

\[
S(m, k) = m^{-1/2} J_m^{-1/2} \sum_{i=m+1}^{m+k} x_i \psi_\tau(y_i - x_i^T \hat{\beta}_i), \quad k = 1, \ldots, T_m,
\]

where \( J_m = \tau (1 - \tau) D_m \) with \( D_m = m^{-1} \sum_{i=1}^{m} x_i x_i^T \), \( \psi_\tau(u) = \tau - 1(u < 0) \) and \( T_m \) is the monitoring horizon. Our proposed test statistic is defined as

\[
Q_{\tau} = \sup_{1 \leq k \leq T_m} \Gamma(m, k, \gamma), \quad \text{where } \Gamma(m, k, \gamma) = \left\| \frac{S(m, k)}{g(m, k, \gamma)} \right\|_{\infty},
\]

\( g(m, k, \gamma) = (1 + k/m)(k/(m + k))^\gamma \) is the normalizing function and \( 0 \leq \gamma < 1/2 \). The tuning parameter \( \gamma \) controls how soon the monitoring will be stopped. The procedure with a larger value of \( \gamma \) will stop sooner and thus is preferred if the structural change occurs shortly after \( m \). Throughout, we call a procedure open-end if the monitoring is continued possibly to infinity if no alarm is raised (that is, the monitoring horizon \( T_m = \infty \)), and closed-end if the monitoring is stopped after a fixed number of observations even if no change is detected (specifically \( T_m/m \to N \) with \( N > 0 \)); see Huskova and Kirch (2012) and Kirch and Kamgaing (2014) for similar definitions.

We propose to stop the monitoring process and reject \( H_0 \) at the stopping time defined by

\[
ST(m) = \begin{cases} 
\inf\{k \geq 1 : \Gamma(m, k, \gamma) \geq c_0\}, & \text{if } \Gamma(m, k, \gamma) < c_0 \text{ for all } k = 1, \ldots, T_m,
\end{cases}
\]

where \( c_0 \) is the critical value chosen to control the false alarm rate at a given significance level \( \alpha \in (0, 1) \), that is, \( \lim_{m \to \infty} P[ST(m) < \infty | H_0] = \alpha \).

We make the following technical conditions.

**Assumption A1.** \((y_1, x_1), (y_2, x_2), \ldots \) are independent random pairs.

**Assumption A2.** The conditional density function of \( y \) given \( x_i \), denoted as \( f_y(\cdot | x_i) \), is continuous, uniformly bounded away from zero and infinity and has a bounded first derivative in the neighborhood of \( x_i^T \beta_{i, \tau} \).

**Assumption A3.** Let \( \| \cdot \| \) denote the Euclidean norm. The sequence \( \{x_i, 1 \leq i < \infty\} \) is strictly stationary satisfying the following conditions:
Similar as in Section 2.1, we also make the "noncontamination" assumption that there exist a historical data of size
\( m \) for each quantile.\footnote{This assumption is made to ensure that the test statistic is not contaminated by the historical data.} There are two cases to consider:

1. There is a change point \( \tau \) such that \( m^{-1} \sum_{i=1}^{m} f_i(x_i, \beta_0') x_i x_i^T \rightarrow D_\tau(\tau) \) as \( m \rightarrow \infty \), and \( k^{-1} \sum_{i=m+1}^{m+k} f_i(x_i, \beta_1') x_i x_i^T \rightarrow D_\tau(\tau) \) as \( k \rightarrow \infty \), where \( D_\tau(\tau) \) represents the distribution of the test statistic under the null hypothesis.

2. There is no change point, and the test statistic converges to \( 0 \) under \( H_0 \).

### Theorem 1
Suppose that Assumptions A1–A3 hold and that \( 0 \leq \gamma < 1/2 \). Then as \( m \rightarrow \infty \), under the null hypothesis \( H_0 \), \( Q_\tau \rightarrow \sup_{0 \leq t \leq 1} \| W(t) \|_\infty / t^\gamma \) for the open-end procedure and \( Q_\tau \rightarrow \sup_{0 \leq t \leq N/(N+1)} \| W(t) \|_\infty / t^\gamma \) for the closed-end procedure, where \( \{ W(t), 0 \leq t < \infty \} \) denotes a \( p \)-dimensional Wiener process.

### Remark 1
By Assumption A1 and the normalization by \( D_m \) in the definition of \( S(m, k) \), it is easy to verify that the components of the limiting process are independent. Based on the asymptotic results in Theorem 1, we can obtain the asymptotic critical values for the proposed test through simulation. We approximate the process \( \| W(t) \|_\infty / t^\gamma \) with \( e_i \sim \text{i.i.d.} N(0, I) \) and \( M = 10000 \). For each replication, the supremum of the approximating process is taken over \( t \in (0, 1) \) if \( T_m = \infty \), over \( t \in (0, N/(N+1)) \) if \( T_m/m > 0 \). In our implementation, the number of replications is 50000. Table 1 summarizes the asymptotic critical values of the open-end procedure at median with \( p = 2 \).

### Table 1
Asymptotic critical values for the open-end procedure at \( \tau = 0.5 \) with \( p = 2 \).

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>0.000</th>
<th>0.025</th>
<th>0.050</th>
<th>0.100</th>
<th>0.250</th>
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<td>3.01</td>
<td>2.73</td>
<td>2.49</td>
<td>2.22</td>
<td>1.82</td>
</tr>
<tr>
<td>0.15</td>
<td>3.06</td>
<td>2.79</td>
<td>2.55</td>
<td>2.29</td>
<td>1.90</td>
</tr>
<tr>
<td>0.25</td>
<td>3.12</td>
<td>2.85</td>
<td>2.62</td>
<td>2.37</td>
<td>1.99</td>
</tr>
<tr>
<td>0.35</td>
<td>3.22</td>
<td>2.96</td>
<td>2.73</td>
<td>2.49</td>
<td>2.12</td>
</tr>
<tr>
<td>0.45</td>
<td>3.47</td>
<td>3.21</td>
<td>3.00</td>
<td>2.78</td>
<td>2.43</td>
</tr>
<tr>
<td>0.49</td>
<td>3.74</td>
<td>3.50</td>
<td>3.29</td>
<td>3.07</td>
<td>2.72</td>
</tr>
</tbody>
</table>

### Theorem 2
Suppose that Assumptions A1–A3 hold and \( 0 \leq \gamma < 1/2 \). Then under \( H_1 \) with \( \delta_m \triangleright 0, \beta_1' - \beta_0' = O(m^{-1/2} a_m) \), where \( a_m \) is an increasing sequence such that \( a_m \rightarrow \infty \), we have \( \sup_{1 \leq k \leq T_m} I(m, k, \gamma) \rightarrow \infty \) as \( m \rightarrow \infty \).

Theorem 2 implies that the proposed test has power tending to one under local alternatives of the order arbitrarily close to \( m^{-1/2} \), which covers the case that \( a_m \rightarrow \infty \) and \( a_m m^{-1/2} \rightarrow 0 \) as \( m \rightarrow \infty \), and that \( \delta = \beta_1' - \beta_0' \) is fixed.

### 2.2. Sequential change point detection across quantiles

We may encounter cases where the main interest is in assessing the stability of the impact of covariates on a range of quantile levels, for instance, the impact of gender on the lower quantiles of salary, or other cases where the quantile subject to structural changes is unknown \textit{a priori}. In both cases, it is desirable to carry out the test across quantiles to monitor the structural changes.

Let \( T = \{ \omega_1, \omega_2 \} \) for some \( 0 < \omega_1 < \omega_2 < 1 \) denote a closed set of quantiles of interest. We assume the following linear quantile regression model
\[
Q_\tau(x_i) = x_i^T \beta_{i, \tau}, \quad \text{for all } \tau \in T, \ i \geq 1.
\]

Similar as in Section 2.1, we also make the "noncontamination" assumption that there exist a historical data of size \( m \) such that \( \beta_{1, \tau} = \cdots = \beta_{m, \tau} = \beta_0' \) for \( \tau \in T \). We are interested in testing the null hypothesis
\[
H_0^*: \beta_{i, \tau} = \beta_0', \quad \text{for all } \tau \in T, \ i \geq m + 1,
\]
against the alternative hypothesis
\[
H_1^*: \beta_{i, \tau} = \begin{cases} 
\beta_0', & \text{for } m + 1 \leq i < m + k^* \\
\beta_1', & \text{for } i \geq m + k^*,
\end{cases}
\]
where \( k^* \geq 1 \) is the unknown change point, and \( \beta_0' \neq \beta_1' \) for some \( \tau \in T \). Extending the idea in Section 2.1, we propose the following test statistic
\[
MQ_{\tau} = \sup_{\tau \in T} \sup_{1 \leq k \leq T_m} \left\| m^{-1/2} D_m^{-1/2} \sum_{i=m+1}^{m+k} x_i \psi_{x_i}(y_i - x_i^T \hat{\beta}_x) / g(m, k, \gamma) \right\|_{\infty}.
\]
Table 2
Type I errors of the proposed closed- and open-end median procedures in Case 1 with \( \gamma = 0 \), 0.25 and 0.4. The nominal significance level is 0.05.

| \( m \) | \( N \) | \multicolumn{3}{c}{Closed-end} | \multicolumn{3}{c}{Open-end} |
|------|------|-------|-------|-------|-------|-------|-------|
|      |      | \( \gamma = 0 \) | \( \gamma = 0.25 \) | \( \gamma = 0.4 \) | \( \gamma = 0 \) | \( \gamma = 0.25 \) | \( \gamma = 0.4 \) |
| 50   | 2    | 0.048 | 0.047 | 0.039 | 0.010 | 0.021 | 0.030 |
|      | 4    | 0.050 | 0.049 | 0.043 | 0.018 | 0.031 | 0.034 |
|      | 20   | 0.054 | 0.052 | 0.045 | 0.045 | 0.047 | 0.042 |
|      | 50   | 0.049 | 0.049 | 0.046 | 0.045 | 0.048 | 0.045 |
|      | 2    | 0.052 | 0.047 | 0.047 | 0.012 | 0.023 | 0.031 |
|      | 4    | 0.049 | 0.047 | 0.044 | 0.020 | 0.028 | 0.035 |
| 200  | 10   | 0.053 | 0.054 | 0.048 | 0.040 | 0.043 | 0.044 |
|      | 20   | 0.054 | 0.053 | 0.051 | 0.043 | 0.048 | 0.049 |
|      | 50   | 0.052 | 0.049 | 0.046 | 0.049 | 0.048 | 0.046 |

Theorem 3. Suppose that \( 0 \leq \gamma < 1/2 \), Assumptions A1 holds, and A2 and A3 hold uniformly in \( \tau \in \mathcal{T} \). Then under the null hypothesis \( H_0^* \), as \( m \to \infty \), \( \text{MQR} \xrightarrow{D} \sup_{\tau \in \mathcal{T}} \sup_{0 \leq t \leq 1} \| W(t, \tau) \| \xrightarrow{\gamma} \sup_{\tau \in \mathcal{T}} \sup_{0 \leq t \leq 1} \| W(t, \tau) \| \xrightarrow{\gamma} \) for the open-end procedure and \( \text{MQR} \xrightarrow{D} \sup_{\tau \in \mathcal{T}} \sup_{0 \leq t \leq N/(N+1)} \| W(t, \tau) \| \xrightarrow{\gamma} \) for the closed-end procedure, where \( W(t, \tau) = \{ W_1(t, \tau), \ldots, W_p(t, \tau) \} \) is a \( p \)-dimensional independent Kiefer process (Kiefer, 1972) with mean 0 and covariance \( E[ W_j(t, \tau), W_{j'}(t', \tau') ] = min(t, t') \{ min(\tau, \tau') - t \tau' \}, \) for \( j = 1, \ldots, p \), which is a Brownian bridge in \( \tau \) and a Wiener process in \( t \).

Remark 2. By Assumption A1 and the definition of \( \text{MQR} \), it is easy to verify that the components of the limiting process are independent. In practice, for calculating \( \text{MQR} \), we approximate the supremum over \( \tau \in \mathcal{T} \) by maximizing over a grid of quantile levels from \( \mathcal{T} \). Similar as in Section 2.1, we can calculate the asymptotic critical values for the multiple-quantile test through simulation. By Lemma 1 in Qu (2008), the process \( \| W(t, \tau) \| \xrightarrow{\gamma} \) can be approximated by \( \| W^*(t, \tau) \| \xrightarrow{\gamma} \), where \( W^*(t, \tau) = \{ W_1^*(t, \tau), \ldots, W_p^*(t, \tau) \} \). \( W_1^*(t, \tau) \) is a Brownian bridge in \( \tau \) and a Wiener process in \( t \).

3. Simulation study

To assess the finite sample performance of the proposed method, we conduct a simulation study. Under \( H_0^* \), data are generated by \( y_i = 1 + x_i + (1 + r x_i) e_i \), \( i = 1, \ldots, m + T_m \), where \( r \) controls the heteroscedasticity of the model. We consider the following four cases: Case 1 with \( r = 0 \), \( e_i \sim N(0, 1) \), \( x_i \sim N(0, 1) \); Case 2 with \( r = 0 \), \( e_i \sim t_2, x_i \sim \text{Uniform}(0, 2) \); Case 3 with \( r = 0.2 \), \( e_i \sim N(0, 1) \), \( x_i \sim N(0, 1) \) truncated at \( \pm 3 \); Case 4 with \( r = 0 \), \( e_i \sim N(0, 1) \), \( x_i \sim \text{Uniform}(0, 1) \). For power analysis, we generate data for Cases 1–3 from the alternative model

\[
y_i = \begin{cases} 
1 + x_i + (1 + \alpha x_i) e_i, & i = 1, \ldots, m + k^* - 1 \\
1 + (1 + \delta) x_i + (1 + \alpha x_i) e_i, & i = m + k^*, \ldots, m + T_m, 
\end{cases} \quad 0 < \delta < 5
\]

and we generate data for Case 4 from

\[
y_i = \begin{cases} 
1 + x_i + (1 - \delta/2) e_i, & i = 1, \ldots, m + k^* - 1 \\
1 + x_i + (1 - \delta/2 + \delta x_i) e_i, & i = m + k^*, \ldots, m + T_m, 
\end{cases} \quad 0 < \delta < 1.5.
\]

We focus on two quantile levels \( \tau = 0.5 \) and \( \tau = 0.75 \). Under the alternative model, in Cases 1–3 a change occurs in the location of the distribution and thus affects both quantiles, while in Case 4 a change occurs in the scale and there is no change point at \( \tau = 0.5 \). We consider \( T_m = Nm \) with \( N = 2, 4, 10, 20 \) and 50 for \( m = 50 \) and 200. For each scenario, the simulation is repeated 3000 times.

3.1. Choice of \( \gamma \)

We first assess the impact of different choices of \( \gamma \) on the detecting procedure. To save space, we only present the results in Case 1 since the other cases show similar phenomena. Table 2 summarizes the Type I errors of the proposed method in Case 1 with \( \gamma \in [0, 0.25, 0.4] \) for both the closed- and open-end procedures at \( \tau = 0.5 \). For all three \( \gamma \) considered, the Type I errors are close to the nominal level of 0.05 for the closed-end procedure. The open-end procedure gives slightly deflated Type I errors for smaller \( N \), but it gives reasonable Type I errors for \( N \geq 20 \), which agrees with the asymptotic theory. Therefore, in practice, we recommend use the closed-end procedure for \( N < 20 \).

Fig. 1 plots the median detection time against \( \gamma \) for the closed-end procedure at \( \tau = 0.5 \). In Case 1 under the alternative model with \( m = 200 \), \( T_m = 2m, \delta = 2 \) and \( k^* = 50 \). Results suggest that larger values of \( \gamma \) tend to give earlier detection, while smaller values of \( \gamma \) lead to more delays of detection. Therefore, a larger value of \( \gamma \) would be preferred in cases where the true structural change occurs shortly after the monitoring begins, and vice versa.
Fig. 1. The median detection time against $\gamma$ for the closed-end procedure at median in Case 1 with $m = 200$, $T_m = 2m$, $k^* = 50$ and $\delta = 2$.

Table 3

Type I errors of the proposed quantile methods at $\tau = 0.5$ and 0.75, and the two mean regression methods XGZ and HHKS in Cases 1–4. The nominal level is 0.05.

<table>
<thead>
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<th>$N$</th>
<th>$m = 50$</th>
<th></th>
<th></th>
<th>$m = 200$</th>
<th></th>
<th></th>
</tr>
</thead>
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<tr>
<td></td>
<td>$\tau = 0.5$</td>
<td>$\tau = 0.75$</td>
<td>XGZ</td>
<td>HHKS</td>
<td>$\tau = 0.5$</td>
<td>$\tau = 0.75$</td>
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<td></td>
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<td></td>
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<tr>
<td>Case 1</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>2</td>
<td>0.047</td>
<td>0.059</td>
<td>0.077</td>
<td>0.032</td>
<td>0.047</td>
<td>0.055</td>
</tr>
<tr>
<td>4</td>
<td>0.049</td>
<td>0.065</td>
<td>0.091</td>
<td>0.030</td>
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<td>0.058</td>
</tr>
<tr>
<td>10</td>
<td>0.045</td>
<td>0.061</td>
<td>0.095</td>
<td>0.032</td>
<td>0.054</td>
<td>0.060</td>
</tr>
<tr>
<td>20</td>
<td>0.052</td>
<td>0.070</td>
<td>0.099</td>
<td>0.032</td>
<td>0.053</td>
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<td>Case 2</td>
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<td>0.059</td>
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3.2. Type I error and power comparison

We now compare the proposed quantile regression method with the mean regression methods developed in Horváth et al. (2004), referred to as HHKS, and in Xia et al. (2009), referred to as XGZ. In this subsection, we let $\gamma = 0.25$ and we consider closed-end procedures for both the quantile regression and HHKS methods.

Table 3 summarizes the Type I errors of the proposed quantile regression method at $\tau = 0.5$ and 0.75, and the mean methods HHKS and XGZ in Cases 1–4, respectively. The XGZ method gives clearly inflated Type I errors across all scenarios. The HHKS method gives inflated Type I errors in Case 2 while deflated Type I errors in the other three cases. The proposed median method controls Type I errors generally better than the other two mean regression methods, especially in Case 2 with a heavy-tailed error distribution. The Type I errors of the quantile method at $\tau = 0.75$ are slightly inflated for $m = 50$ but they are close to the nominal level when $m$ increases to 200.
Fig. 2. The power curves of the quantile regression method at $\tau = 0.5$ (long-dashed), $\tau = 0.75$ (solid), the XGZ (dash-dotted) and HHKS (short-dashed) methods against $\delta$ in Cases 1–4 with $m = 200$, $T_m = 2m$ and $k^* = 50$.  

Fig. 2 plots the power curves of different methods against the slope change $\delta$ in Cases 1–4 with $m = 200$, $T_m = 2m$ and $k^* = 50$. In Cases 1 and 3 with normal errors, the XGZ method has similar power with the quantile regression method even though the former is associated with larger Type I errors, while the HHKS method is overall too conservative. When the error distribution has heavy tails (Case 2), the proposed method at both quantiles shows higher power than the two mean methods as $\delta$ increases despite the much larger Type I errors of the latter two methods. In Case 4 with $\delta \neq 0$, changes occur in the tail of the distribution but not in the center; the proposed test procedure is powerful at $\tau = 0.75$ and it has no power (as expected) at $\tau = 0.5$. One interesting finding is that in Case 4 with $\delta \neq 0$, there is no change in the conditional mean function, but both the XGZ and HHKS methods tend to reject the null hypothesis and thus have high false positives. In contrast, the proposed quantile regression method is able to capture the heterogeneity and distinguish structural changes at different quantiles of the conditional response distribution.

Acknowledgments

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Appendix A. Supplementary data

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References