Nuclear Physics: Conventions

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The Natural System of Units is particularly popular in Nuclear and High-Energy Physics since as many fundamental constants as possible have as simple a value as possible (see [MM]!).
Set the speed of light and Planck's quantum to $c = \hbar = 1$. This expresses velocities in units of c, and actions and angular momenta in units of $\hbar$. Then, only one fundamental unit remains, namely either an energy- or a length-scale. Time-scales have the same units as length-scales. We also set Boltzmann's constant $k_\mathrm{B} = 1$, so energy and temperature have the same units. Now one only memorises a handful of numbers. [Setting Newton's gravitational constant $G_N = 1$ eliminates *any* dimension-ful unit – only String Theorists do that.]

Electrodynamic Units: The Rationalised Heaviside-Lorentz system will be used throughout. Formally, it can be obtained from the SI system by setting the dielectric constant and permeability of the vacuum to $\epsilon_0 = \frac{1}{\mu_0 c^2} = 1$. The system is uniquely determined by any two of the fundamental equations which contain $\vec{E}$ and a combination of $\vec{E}$ and $\vec{B}$. More on systems, units and dimensions e.g. in [MM].

Charges $Q = Ze$ are measured in units Z of the elementary charge $e > 0$; electron charge $-e < 0$.
Lagrangean: $\mathcal{L}_\mathrm{elmag} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \implies$ Maxwell's equations: $\partial_\mu F^{\mu\nu} = j^\nu$
Lorentz force: $\vec{F}_L = Ze[\vec{E} + \vec{\beta} \times \vec{B}]$; Coulomb's law: $\Phi(r) = \frac{Ze}{4\pi\, r}$

"Restoring" SI units from "natural units": Multiply by $c^\alpha\, \hbar^\beta\, k_\mathrm{B}^\gamma\, \epsilon_0^\delta$ and determine the exponents such that the proper SI unit remains, using $[c]$: $[\mathrm{m\,s^{-1}}]$, $[\hbar]$: $[\mathrm{kg\,m^2\,s^{-1}}]$, $[k_\mathrm{B}]$: $[\mathrm{m^2\,kg\,s^{-2}\,K^{-1}}]$ and $[\epsilon_0]$: $[\mathrm{C\,V^{-1}\,m^{-1}} = \mathrm{C^2\,s^2\,m^{-2}\,kg^{-1}}]$. Example: $E = m \implies E = m\, c^\alpha\, \hbar^\beta\, k_\mathrm{B}^\gamma\, \epsilon_0^\delta$, and you have to convert $\mathrm{kg\,m^2/s^2}$ into kg, i.e. add two powers of m/s, so that $\alpha = 2, \beta = \gamma = \delta = 0$.

Conventions Relativity: Einstein Σummation Convention; "East-coast" metric $(+\,-\,-\,-)$:
$A^2 \equiv A^\mu\, A_\mu := (A^0)^2 - \vec{A}^2$. Velocity β, Lorentz factor $\gamma = \left(1 - \beta^2\right)^{-1/2}$.

Conventions QFT: "Bjørken/Drell": [HM, PS] – close to [HH], but fermion norms different:

Quantised complex scalar: $\Phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left[a(\vec{k})\, \mathrm{e}^{-\mathrm{i}k\cdot x} + b^\dagger(\vec{k})\, \mathrm{e}^{+\mathrm{i}k\cdot x}\right]$ with $E_k := k^0 = +\sqrt{\vec{k}^2 + m^2}$

Minimal substitution in QED: $D^\mu = \partial^\mu + \mathrm{i}Ze\, A^\mu$; in non-Abelian gauge theories (QCD,...): $D^\mu = \partial^\mu - \mathrm{i}g\, A^\mu$.

$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5$; $2mP_+ := \sum\limits_{s=\pm} u_s(p)\bar{u}_s(p) = \not{p} + m,\ -2mP_- := \sum\limits_{s=\pm} v_s(p)\bar{v}_s(p) = \not{p} - m \implies P_+ + P_- = 1$

Elastic cross section (our convention) in cm: $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\, |\overline{\mathcal{M}}|^2$; lab, $m = 0$: $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 M^2} \left(\frac{E'}{E}\right)^2 |\overline{\mathcal{M}}|^2$

Decay of particle with mass M (cm, our convention): $\Gamma[A \to B(\vec{p}') + C] = \frac{|\vec{p}'|}{8\pi M^2} \int \mathrm{d}\Omega\, |\overline{\mathcal{M}}|^2$

More cross section formulae & conventions in "**Summary Electron Scattering Cross Sections**" below.

Nuclear Physics: Some Oft-Overlooked Bare Essentials

Know these by heart! Physicists spend a lot of time solving complicated problems, so we want to always start with an idea of the result. We have ideas when we are hiking, cycling, in the shower, etc., and usually not on our desk. We discuss them with colleagues on the blackboard, and we cannot waste their and our time with looking stuff up. Therefore, we need to be able to do calculations without computers, books or calculators, i.e. in our head or with a piece of scrap paper and a dull pencil.
Here a list of numbers most commonly used for estimates, back-of-the-envelope calculations, etc. of the *Nuclear Physics tool-chest. You need them for that purpose but they are often overlooked.* You should know the following by heart when woken up at night. The list is not exhaustive, not meant to be relevant and may not be useful. Your mileage may vary. It's better to know too much than not enough. You should check these values, how they come about, and their limitations. Trivialities are not included, like most "Essentials' from Mathematical Methods [MM] (dimensional estimates, guesstimates, Natural System of Units). If you know more things worth remembering, let me know!
This is the list of *often-overlooked numbers*, *not of the minimum necessary* – **the minimum is much larger** (including names, spins, charges, masses of all fundamental particles, etc.).

Please turn over.

$\approx$: number rounded for easier memorising – it suffices to know the first significant figure.
$\widehat{=}$, $\widehat{\approx}$: correspondences are correct only in the natural system of units.

Quantity	**Value**
speed of light in vacuum	$c := 2.997\,924\,58 \times 10^8$ m s^{-1}(def.!)$\approx 3 \times 10^8$ m s^{-1}
energy electron gains when accelerated by 1 V	1 electron Volt (eV) $\approx 1.6 \times 10^{-19}$ Joule $= 1\frac{e}{[\mathrm{C}]}$ J
conversion factor	1 J $\approx 6 \times 10^{18}$ eV
typ. subatomic length-scale (proton/neutron size)	1 fermi (femtometre, fm) $= 10^{-15}$ m
conversion factor energy – length	$\hbar c \widehat{=} 1 = 197.327\ldots$ MeV fm ≈ 200 MeV fm
$\Longrightarrow$ conversion factor distance – time (nat. units) (time for light to travel a typ. distance-scale)	$1\text{ fm} = 1\frac{\text{fm}}{c} \widehat{\approx} \frac{1}{3} \times 10^{-23}$ s
conversion factor elmag.: fine-structure constant el. strength at atomic/nuclear/hadronic scales	$\alpha = \left.\frac{e^2}{4\pi}\right\rvert_{\text{nat.+rat. HL}} = \left.\frac{e^2}{4\pi\epsilon_0\hbar c}\right\rvert_{\text{SI}} \approx \frac{1}{137}$ (no units!)
conversion factor energy – temperature: $E = k_{\mathrm{B}}T$	1 eV $\widehat{\approx}$ 11 600 Kelvin, 300 K $\widehat{\approx} \frac{1}{40}$ eV
"classical" electron radius	$r_{\mathrm{e}} = \frac{\alpha}{m_{\mathrm{e}}c^2} = \left.\frac{e^2}{4\pi\, m_{\mathrm{e}}}\right\rvert_{\text{nat.+rat. HL}} \approx 3$ fm

Masses conversion factor: atomic unit $1\text{u} = \frac{\text{mass }^{12}\text{C atom}}{12} = \frac{1}{12} \times \frac{12\text{ g}}{6.022 \times 10^{23}} \approx \frac{1}{6} \times 10^{-23}$ g

electron	$m_{\mathrm{e}} \approx 511$ keV	muon	$m_\mu \approx 110$ MeV $\approx 200\, m_{\mathrm{e}}$	
nucleon	$M_{\mathrm{N}} \approx 940$ MeV $\approx 1800\, m_{\mathrm{e}} \approx 1$ GeV ≈ 1 u			
proton	$M_{\mathrm{p}} \approx 938$ MeV$= 6\pi^5\, m_{\mathrm{e}}$	neutron	$M_{\mathrm{n}} \approx 940$ MeV	$\Longrightarrow$ p-n mass difference 1.3 MeV $\approx 3\, m_{\mathrm{e}}$
pion	$m_\pi \approx 140$ MeV $\approx \frac{1}{7}\, M_{\mathrm{N}}$	kaon	$m_{\mathrm{K}} \approx 500$ MeV	ρ, ω mesons $m_\rho \approx m_\omega \approx 800$ MeV
Higgs boson	$M_{\mathrm{H}} \approx 125$ GeV	W boson	$M_{\mathrm{W}} \approx 80$ GeV	Z boson $M_{\mathrm{Z}} \approx 90$ GeV

Scattering nuclear cross-section unit: 1 barn $b = 100\text{ fm}^2 = (10\text{ fm})^2 = 10^{-28}\text{ m}^2 \approx \frac{1}{400\text{ MeV}^2}$

"geometric" scattering: $\sigma_{\text{geometric}} = 4\pi\, a^2$.

Interpretation: (1) class. point particle on sphere, radius a, any energy; (2) QM zero-energy, scatt. length a.

Hierarchy of Scales	typ. energy	typ. momentum	typ. size/distance
nuclear structure	binding: 8MeV per nucleon	100 keV... 1MeV	10fm ($\sim^{235}$U size)
few-nucleon	binding: deuteron: 2.2246MeV; ^{4}He: 24MeV	$m_\pi \approx 140$MeV	$\frac{1}{m_\pi} \approx 1.5$fm (Yukawa)
hadronic	$M_{\mathrm{N}}, M_\rho \approx 1$GeV	1GeV (relativistic)	$\frac{1}{M_{\mathrm{N}}} \approx 0.2$fm
particle	100GeV Z, W masses	100GeV (relativistic)	$\frac{1}{100\text{GeV}} \approx 2 \times 10^{-3}$fm

Interaction Scales very rough – factors of 100 up or down are common

	strong (NN int.)	**strong** (qq int.)	**elmag**	**weak** (nuclear)	**weak** (hadronic)
range	$\frac{1}{m_\pi} \approx 1.4$ fm	$\frac{1}{1\text{ GeV}} \approx 0.2$ fm	∞	$\frac{1}{M_{\mathrm{W,Z}}} \approx 0.01$ fm	
life time τ	10^{-22} s	10^{-23} s	10^{-20} s	10^{-10} s	10^{-9} s
decay width Γ	200 MeV		1 MeV	$\ll$ eV	
cross section σ	barn (NN@1MeV: 70b)	mb	μb	10^{-12} b $= 1$ pb	100 pb

Miscellaneous neutron lifetime: $\tau_{\mathrm{n}} \approx 880$ s hadron size $R \approx 0.7$ fm hadronisation scale 1 GeV $\widehat{\equiv}$ 0.2 fm

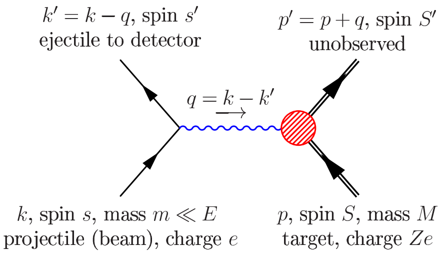
Weinberg mixing angle $\sin^2\theta_{\mathrm{W}} \approx 0.22 \approx 1 - \frac{M_{\mathrm{W}}^2}{M_{\mathrm{Z}}^2}$ Fermi constant $G_{\mathrm{F}} \approx \frac{\sqrt{2}\, g^2}{8M_{\mathrm{W}}^2} \approx 1 \times 10^{-5}$ GeV^{-2}

running coupling constants: $\alpha(1\text{ MeV}) = \frac{1}{137}$; $\alpha(M_{\mathrm{Z}}) = \frac{1}{128}$ $\alpha_{\mathrm{s}}(2m_{\mathrm{b}} \approx 10\text{ GeV}) \approx 0.2$; $\alpha_{\mathrm{s}}(M_{\mathrm{Z}}) \approx 0.118$

Summary Electron Scattering Cross Sections

cf. [HM 8]

Lowest-order Feynman graph



Mandelstam: $s = (k+p)^2$; $t = (k'-k)^2$; $u = (p'-k)^2$

$k = (E, \vec{k})$, $k' = (E', \vec{k}')$, $q^2 = -Q^2 < 0$

$M\nu = p \cdot q = M(E' - E)_{\text{lab}}$

scatt. angle θ: $\vec{k} \cdot \vec{k}' = |\vec{k}| \cdot |\vec{k}'| \cos\theta$

for elastic in lab frame $\quad \dfrac{E'}{E} = \dfrac{1}{1 + \frac{E}{M}(1 - \cos\theta)}$

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{cm}} = \frac{1}{64\pi^2 s}\,|\overline{\mathcal{M}}|^2$$ **elastic, unpolarised, cm-frame**

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{lab}} = \frac{1}{64\pi^2 M^2}\left(\frac{E'}{E}\right)^2 |\overline{\mathcal{M}}|^2$$ **elastic, unpolarised, lab-frame**

$$|\overline{\mathcal{M}}|^2 = (e^2)^2 L_{\mu\nu}\,\frac{1}{q^4}\,W^{\mu\nu}$$ **matrix element**: avg. initial, sum final spins

$$e^2 L^{\mu\nu} = \frac{1}{2s+1}\sum_{s,s'} j^\mu_{s,s'}(k,k')\, j^{\nu\dagger}_{s,s'}(k,k')$$ **lepton tensor**: scatter off virtual γ

$$j^\mu_{s,s'}(k,k') = \langle k', s' | j^\mu | k, s\rangle$$ **lepton current** (electromagnetic)

$$j^\mu_{s,s'} = -\mathrm{i}e\,\bar{l}_{s'}(k')\,\gamma^\mu\, l_s(k)$$ for electron

$$L^{\mu\nu} = 2[k^\mu k'^\nu + k'^\mu k^\nu - g^{\mu\nu}\, k \cdot k']$$ for electron ($m = 0, s = \frac{1}{2}$); cf. (I.7.4W)

$$e^2 W^{\mu\nu} = \frac{1}{2S+1}\sum_{S,S'} J^\mu_{S,S'}(p,p') J^{\nu\dagger}_{S,S'}(p,p')$$ **hadron tensor**: scatter off virt. γ

$$J^\mu_{S,S'}(p,p') = \langle p', S' | J^\mu | p, S\rangle$$ **hadron current** (electromagnetic)

$$q_\mu j^\mu = 0 = q_\mu J^\mu \implies q_\mu L^{\mu\nu} = 0 = q_\mu W^{\mu\nu}$$ **elmag. current conservation**

Relativistic Rutherford: Coulomb of massless, spin-0 on infinitely heavy, spin-0 point-target ($s = 0; M = \infty, S = 0$)

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{lab}} = \left(\frac{Z\alpha}{2E\sin^2\frac{\theta}{2}}\right)^2 \qquad J^\mu = -\mathrm{i}Ze\delta^{\mu 0}\text{, i.e. point charge at rest} \tag{I.7.1}$$

(I.7.1C)

$e^\pm$ **Coulomb scattering on infinitely heavy, composite spin-0 target:** ($s = \frac{1}{2}; M = \infty, S = 0$)

helicity forbids back-scattering; isotropic charge density $\rho(r)$

$$\left(\frac{Z\alpha}{2E\sin^2\frac{\theta}{2}}\right)^2 \cos^2\frac{\theta}{2} \times |F(\vec{q}^2)|^2 \quad \overset{\text{Fourier}}{\Longrightarrow} \text{ form factor } F(\vec{q}^2) := \frac{4\pi}{Ze}\int_0^\infty dr\, \frac{r}{q}\,\sin(qr)\,\rho(r) \tag{I.7.2}$$

normalisation: $F(0) = 1$; charge radius: $\langle r^2\rangle = -6\left.\dfrac{dF(\vec{q}^2)}{d\vec{q}^2}\right|_{\vec{q}^2=0}$

$e^\pm$ **full electromagnetic scattering on massive composite spin-0 target:** ($s = \frac{1}{2}; M$ finite, $S = 0$)

$$\underbrace{\left(\frac{Z\alpha}{2E\sin^2\frac{\theta}{2}}\right)^2 \cos^2\frac{\theta}{2}\,\frac{E'}{E}}_{\textbf{Mott: } \text{no structure}} \times |F(q^2)|^2$$

adds target recoil; $\vec{q}^2 \to q^2$: 4-momentum transfer (I.7.3)

$J^\mu = -\mathrm{i}Ze\, F(q^2)\,(p^\mu + p'^\mu)$ (most general for $S = 0$) (I.7.3C)

$\Longrightarrow W^{\mu\nu} = Z^2 (p+p')^\mu\,(p+p')^\nu\,|F(q^2)|^2$ (I.7.3W)

$e^\pm\mu^\pm \to e^\pm\mu^\pm$ **scattering on massive spin-$\frac{1}{2}$ target without structure:** ($s = \frac{1}{2}; M, S = \frac{1}{2}$)

$$\left(\frac{Z\alpha}{2E\sin^2\frac{\theta}{2}}\right)^2 \cos^2\frac{\theta}{2}\,\frac{E'}{E} \times \left[1 - \frac{q^2}{2M^2}\,\tan^2\frac{\theta}{2}\right] \tag{I.7.4}$$

back-scattering by helicity transfer (spin-spin/mag. moment interaction)

Massive $S = \frac{1}{2}$ hadr. tensor, no structure: $\quad W^{\mu\nu} = 2Z^2\left[p^\mu p'^\nu + p'^\mu p^\nu - g^{\mu\nu}(p \cdot p' - M^2)\right]$ (I.7.4W)

$e^\pm$ **on composite, massive spin-$\frac{1}{2}$ target:** form factors $F_1(q^2)$: Dirac; $F_2(q^2)$: Pauli ($s = \frac{1}{2}; M, S = \frac{1}{2}$)

$$\left(\frac{\alpha}{2E\sin^2\frac{\theta}{2}}\right)^2 \cos^2\frac{\theta}{2}\,\frac{E'}{E} \times \left[\left[F_1^2(q^2) + \tau F_2^2(q^2)\right] + 2\tau\,\left[F_1(q^2) + F_2(q^2)\right]^2\,\tan^2\frac{\theta}{2}\right] \qquad \tau := -\frac{q^2}{4M^2} \tag{I.7.5}$$

Variant: **Rosenbluth/Sachs formula** uses **Sachs form factors** $G_E = F_1 - \tau F_2$, $G_M = F_1 + F_2$

$$= \qquad [\dots] \qquad \times \left[\frac{G_E^2(q^2) + \tau\, G_M^2(q^2)}{1+\tau} + 2\tau\, G_M^2(q^2)\,\tan^2\frac{\theta}{2}\right] \tag{I.7.5}$$

$$J^\mu = -\mathrm{i}\underbrace{e\,F_1(q^2)\,\bar{u}(p')\gamma^\mu u(p)}_{\substack{\text{modify point form}\\ F_1(0) = Z = G_E(0)}} + \frac{e}{2M}\underbrace{F_2(q^2)\,q_\nu\,\bar{u}(p')\mathrm{i}\sigma^{\mu\nu}u(p)}_{\substack{\text{anomalous mag. term, } F_2(0) = \kappa = G_M(0) - Z;\\ \kappa \text{ anomalous mag. moment, } \mu = Z + \kappa \text{ mag. moment}}} \quad (\text{most general for } S = \tfrac{1}{2}) \tag{I.7.5C}$$

$e^\pm$ ***inelastic*, inclusive scattering:** E' independent variable, p' not detected ($s = \frac{1}{2}; M, S =$ any [sic!])

$$\left.\frac{d\sigma}{d\Omega\, dE'}\right|_{\text{lab}} = \left(\frac{2\alpha}{q^2}\right)^2 E'^2 \cos^2\frac{\theta}{2} \times \left[\frac{F_2(q^2,x)}{\nu} + \frac{2\,F_1(q^2,x)}{M}\,\tan^2\frac{\theta}{2}\right] \tag{I.7.6}$$

inelasticity measure: **Bjorken-**$x := -\dfrac{q^2}{2p \cdot q} = \dfrac{Q^2}{2M\nu} \in [0;1]$; elastic: $x = 1$, i.e. $F_{1,2}(q^2,x) \propto \delta(\nu + \dfrac{q^2}{2M})$

Most general elmag. hadronic ME (symmetric $\mu \leftrightarrow \nu$, charge conservation; any spin [sic!]):

$$W^{\mu\nu} = \frac{F_1(q^2,x)}{M}\left[\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right] + \frac{F_2(q^2,x)}{M^2\nu}\left[p^\mu - \frac{p \cdot q}{q^2}\,q^\mu\right]\left[p^\nu - \frac{p \cdot q}{q^2}\,q^\nu\right] \tag{I.7.6W}$$

Structure functions F_1, F_2 *are not the Dirac, Pauli FFs of Eq.* (I.7.5)!