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III. Descriptions

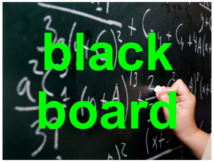
2. Perturbative QCD

Or: Why we Believe

References: [PRSZR 8.1-3, 14; HM 2.15, 10.3-9, 11.4/6-7; Tho 10.7/8; Ryd 3, end of 9.6; HG 12.3; PS 16.7; Per 6.5; lots more. . .]

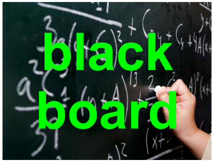
(a) An Ideal World: QCD With Small Coupling Constant

(b) From Colours to Potentials



(c) Running Coupling & Asymptotic Freedom

QED: [Ryd, end of 9.6]
QCD: [PS 16.7, Per 6.5]



Running Coupling in QCD: Now Known to $\mathcal{O}(\alpha_s^4)\widehat{=}3$-Loop

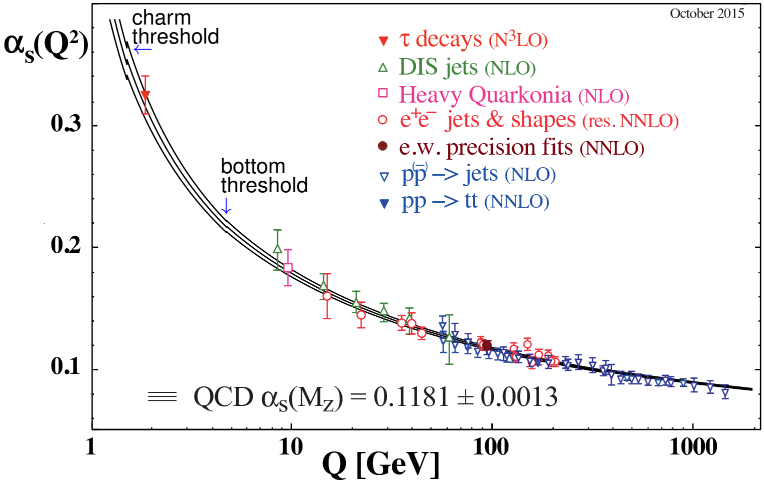
$SU(N_c)$ Gauge Theory at LO (1-loop)

N_f quark flavours with $m_q^2 < q^2$: $\alpha_s(q^2) = \dfrac{4\pi}{[11N_c - 2N_f]\ln(|q^2|/\Lambda^2_{\text{QCD}})}$ (for $m_q = 0$)

[Gross, Politzer/Wilczek, 't Hooft 1973]

Today calculated up to & including $\mathcal{O}(\alpha_s^3)$ relative to LO: horrific diagrams, beautifully agrees with data.

$\Longrightarrow$ **QCD has one parameter. Data fit:** $\alpha_s(M_z) = 0.1181 \pm 0.0013$ **or** $\Lambda_{\text{QCD}} \approx 250$ MeV.

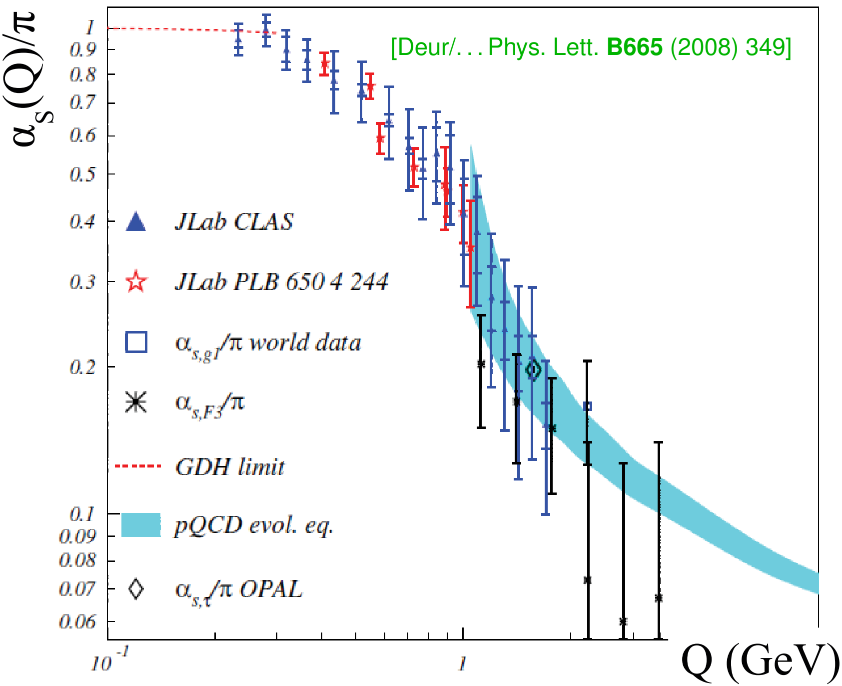


[PDG 2015]

This Confirms:
- perturbative renormalisation procedure
- gauge group is $SU_c(N_c)$, and $N_c = 3$
- flavour $N_f = (uds) + (c) + (b)$ increases like R-factor with $\sqrt{|q^2|}$

The Low-$|q^2|$ Regime: Infrared Slavery

$$\alpha_s(q^2) = \frac{4\pi}{[11N_c - 2N_f]\ln(|q^2|/\Lambda^2_{\rm QCD})} + \mathcal{O}(\alpha_s^3)$$



Naïvely apply running $\Longrightarrow$

$\alpha_s > 1$ at $\sqrt{|q^2|} \approx 1\text{GeV}$

$\Longrightarrow$ Perturbation theory breaks down at low $|q^2|$.

$\Longrightarrow$ Must resort to **non-perturbative** methods!

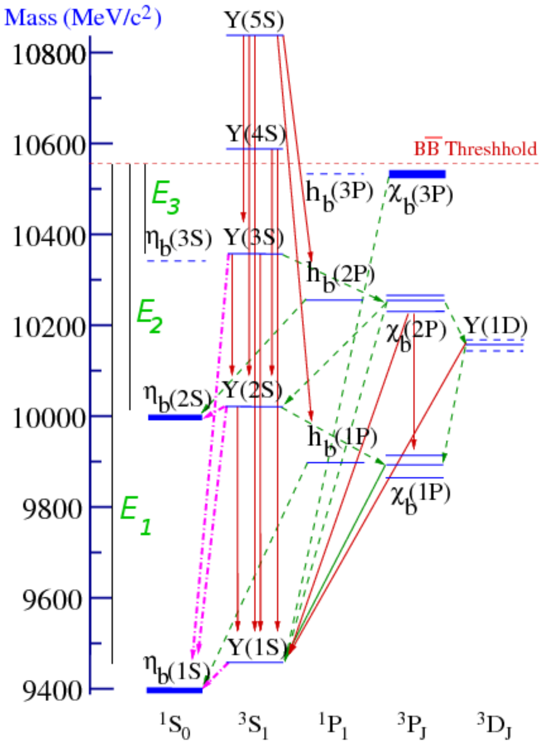
Infrared Slavery
offers *chance*
of confinement.

Is typical size of charge-smearing set by $\dfrac{1}{\Lambda_{\rm QCD} \approx 250\text{MeV}} \approx 1\text{fm}$?

$\Longrightarrow$ Hadron size, confinement?

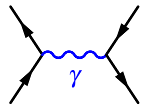
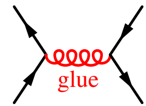
(d) Quarkonia and Perturbative QCD

LO QCD $\hat{=}$ QED$^{N_c^2-1}$ for $\alpha_s(q^2) \ll 1$ $\Longrightarrow$ Test on positronium-like $q\bar{q}$ at large $|q^2|$.



[Bottomonium spectrum($b\bar{b}$).]

Positronium: H-atom with *reduced mass* $\mu = \frac{m_e}{2}$

	positronium e^+e^-	quarkonium $q\bar{q}$
pot. $V(r)$	$-\frac{\alpha}{r}$ 	$-\frac{4}{3}\frac{\alpha_s}{r}$ 
binding E_n	$-\alpha^2 \frac{\mu_e}{2n^2}$	$-\left(\frac{4}{3}\alpha_s\right)^2 \frac{\mu_q}{2n^2}$

Should work best for heaviest system: **Bottomonium $b\bar{b}$**

$\Longrightarrow$ If truly Coulombic, then $\frac{E_1 - E_2}{E_2 - E_3} = \frac{1 - \frac{1}{2^2}}{\frac{1}{2^2} - \frac{1}{3^2}} = \frac{27}{5}$.

$\Longrightarrow$ Long-range part not really Coulombic!

$\Longrightarrow$ **Add *phenomenological* QCD String Potential**

$$V(r) = -\frac{4\alpha_s}{3r} + \sigma\, r$$

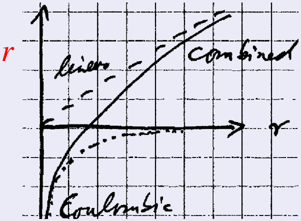
String constant $\sigma \approx 1\,\frac{\text{GeV}}{\text{fm}} \approx 10^5\,\frac{\text{N}}{\text{fm}}$

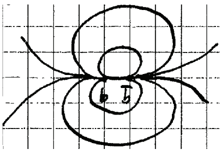
by fit to spectra; seems universal in $b\bar{b}, c\bar{c}$.

***Phenomenological* QCD String Potential**: $V(r) = -\frac{4\alpha_s}{3r} + \sigma\, r$

Coulomb $+$ String with **String constant** $\sigma \approx 1\,\frac{\text{GeV}}{\text{fm}} \approx 10^5\,\frac{\text{N}}{\text{fm}}$.

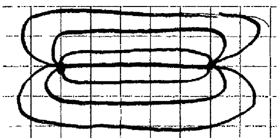
"Best of all simple modifications."





Short-distance: Coulombic dominates: $\frac{4\alpha_s}{3}\left[\frac{1}{|\vec{r}-\vec{a}|} - \frac{1}{|\vec{r}+\vec{a}|}\right]$

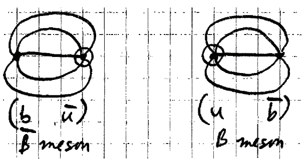
$\Longrightarrow$ Colour-electric dipole between opposite colour charges.



Medium distance: Dipole field gets "squeezed" into **colour-flux tube**, like potential of rubber band/string.

$\Longrightarrow$ Energy of states "pushed up" relative to Coulomb.

Larger distance: $V(r \to \infty)$ increases linearly. $\Longrightarrow$ String narrows, infinite energy to isolate quark, consistent with **Confinement Hypothesis**.



$r \approx 1\text{fm}$**:** String stores energy $\sigma\, r \approx 1\text{GeV}$.

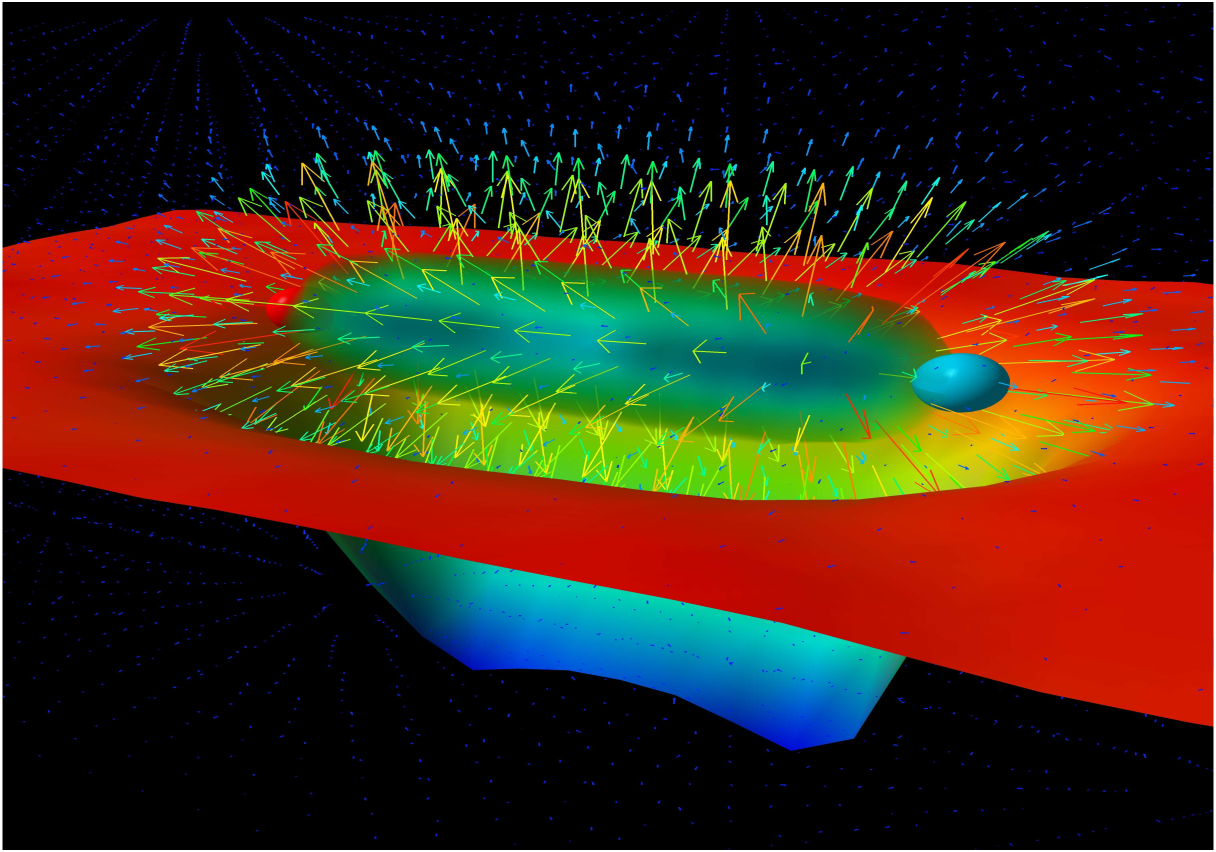
$\Longrightarrow$ Suffices to produce light $q\bar{q}$ pairs from vacuum *and* hadronise them:

String Breaking at **Hadronisation/Fragmentation Scale** 1fm.

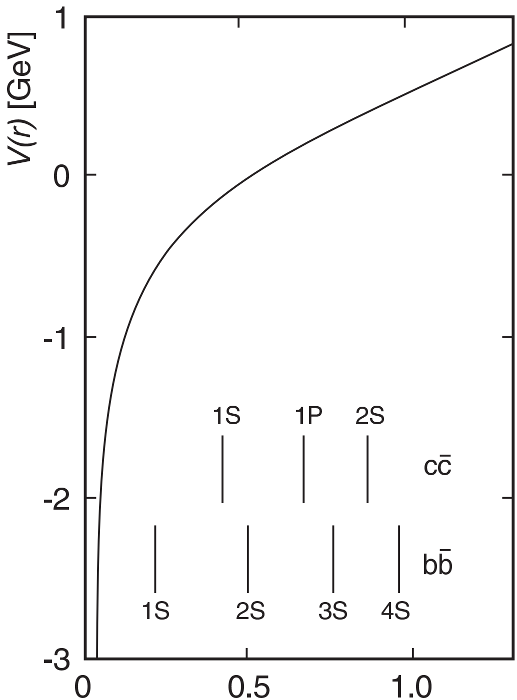
$$b\bar{b}(9,400) \to [(b\bar{u}) + (u\bar{b})]\ (2\times 5,300 = 10,600):\ \Delta M \approx 1\text{GeV}\checkmark$$

Actual (Lattice) QCD: Heavy-Meson Flux Tube

[Leinweber et al. 2003, click here for homepage]



Phenomenological Potentials: Constituent Quark Model



[PRSZR]

– **Take** perturbative QCD results for colour factors etc.

– **Fit** string constant σ, quark (constituent) mass m_q, α_s.

– **Non-relativistic potential** with some retardation effects:

HFS $\dfrac{8\pi\alpha_s\,\vec{\sigma}_1\cdot\vec{\sigma}_2}{9\,m_q^2}\delta^{(3)}(\vec{r})$, FS $\propto \vec{L}\cdot\vec{S}$, Darwin, Lamb,...

Results Bottom: $m_b \approx 5\text{GeV}$, $\alpha_s(\Upsilon) \approx 0.2$, $\sigma_\Upsilon \approx 1\,\frac{\text{GeV}}{\text{fm}}$

Charm: $m_c \approx 1.5\text{GeV}$, $\alpha_s(J/\psi) \approx 0.25$, $\sigma_{J/\psi} \approx 1\,\frac{\text{GeV}}{\text{fm}}$

– **QCD string constant same for b and c: universal**

– Constituent quark masses of b and c slightly larger than their QCD (current quark) masses: **small "dressing"**.

– **Charmonium** less Coulombic; more relativistic; more sensitive to QCD string.

– Confirms perturbative colour factors. $\Longrightarrow SU(N_c = 3)$.

– **But usually HFS somewhat small, LS somewhat big.**

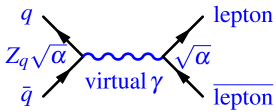
Neglects many relativistic radiative/retardation effects.

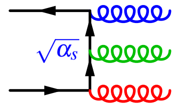
QCD-inspired Constituent Quark *Model* was important to boost confidence in QCD.

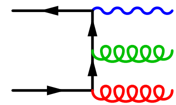
– Now we need to go beyond and do "true" QCD!

Actual QCD: Determine $\alpha_s(q^2)$ in $e^+e^- \to \gamma^*(J^{PC}{=}1^{--}) \to$ Quarkonium

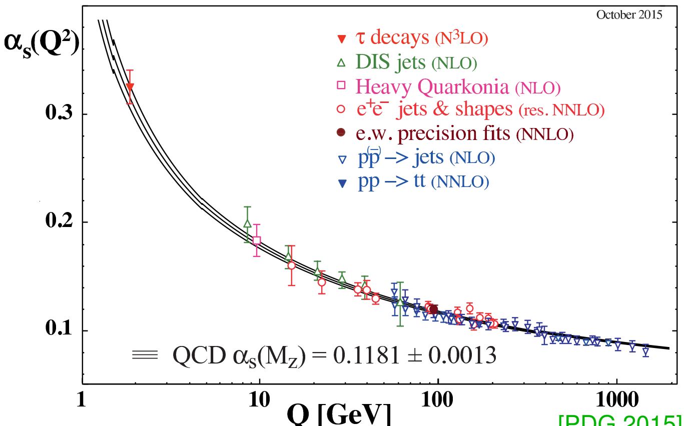
J/ψ and Υ are $^3S_1$ states: $\Longrightarrow$ Only decay into odd number of gauge bosons (parity, see HW).







$$\Longrightarrow \frac{\Gamma[\text{leptons}]}{\Gamma[\text{3jets}]} \propto \frac{(Z_q\alpha)^2}{\alpha_s^3} \quad ; \quad \frac{\Gamma[\text{leptons}]}{\Gamma[\gamma+\text{2jets}]} \propto \frac{(Z_q\alpha)^2}{Z_q^2\alpha\alpha_s^2} = \frac{\alpha}{\alpha_s^2} \quad ; \quad \frac{\Gamma[\text{3jets}]}{\Gamma[\gamma+\text{2jets}]} \propto \frac{\alpha_s^3}{Z_q^2\alpha\alpha_s^2} = \frac{\alpha_s}{Z_q^2\alpha}$$



[PDG 2015]

Include QCD corrections to high orders.

Lots of experimental information,

many $b\bar{b}$ states & decays not yet seen.

$$\alpha_s(\Upsilon) = 0.163 \pm 0.016$$
$$\alpha_s(J/\psi) = 0.25 \pm 0.05$$

But only one datum on plot.

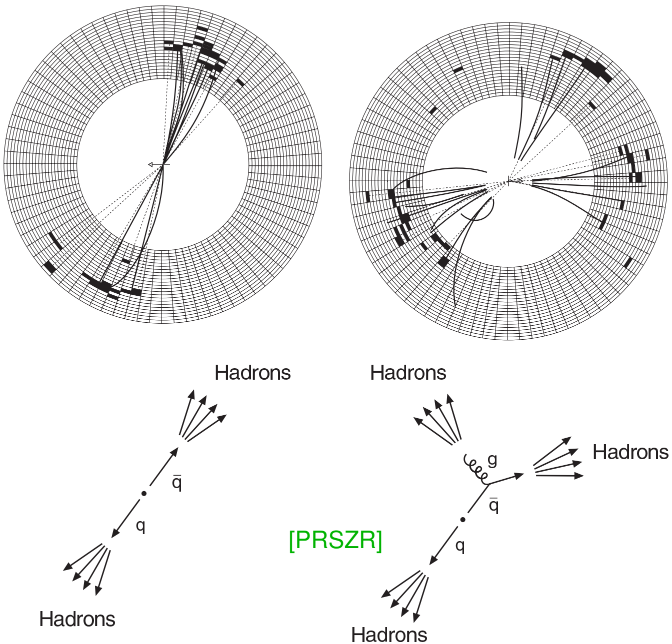
$\Longrightarrow$ Can do even better.

(g) Perturbative QCD Corrections in e^+e^- Annihilation

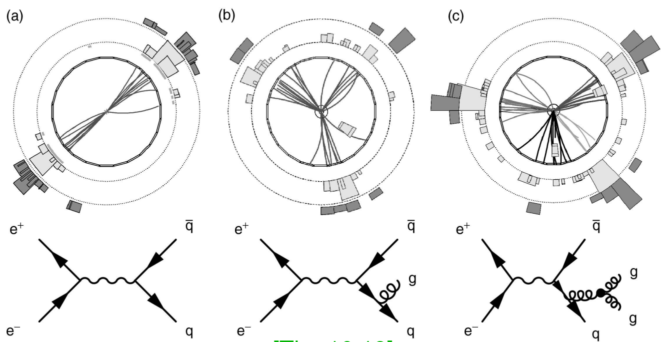
2 & 3 Jet Events: Evidence of Gluons at Large $\sqrt{s}$

PETRA 1979

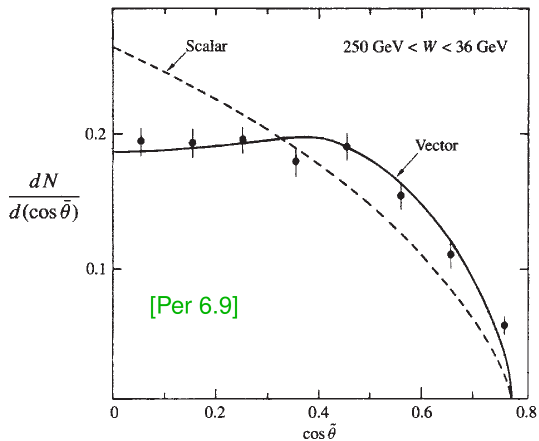
If third jet, its total charge is often *zero*. Ratio $\frac{\text{3 jets}}{\text{2 jets}} \simeq \alpha_s(s) < 1$ for large $\sqrt{s}$.



[PRSZR]



[Tho 10.19]



[Per 6.9]

3-Jet Events: angular distribution tests gluon spin, parity, charge-conjugation: $J^{PC} = 1^{--}$.

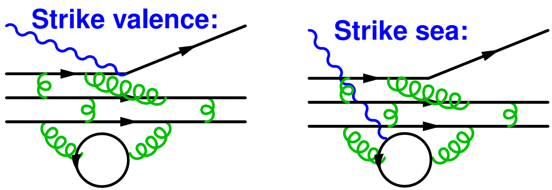
You could calculate this with what we learned.

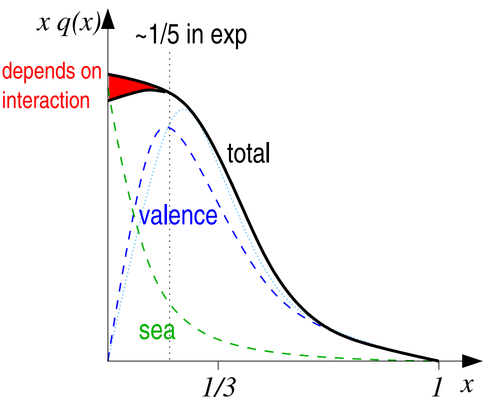
4-Jet Events: test ggg vertex $\Longleftrightarrow$ **local $SU(3)$ gauge symmetry**.

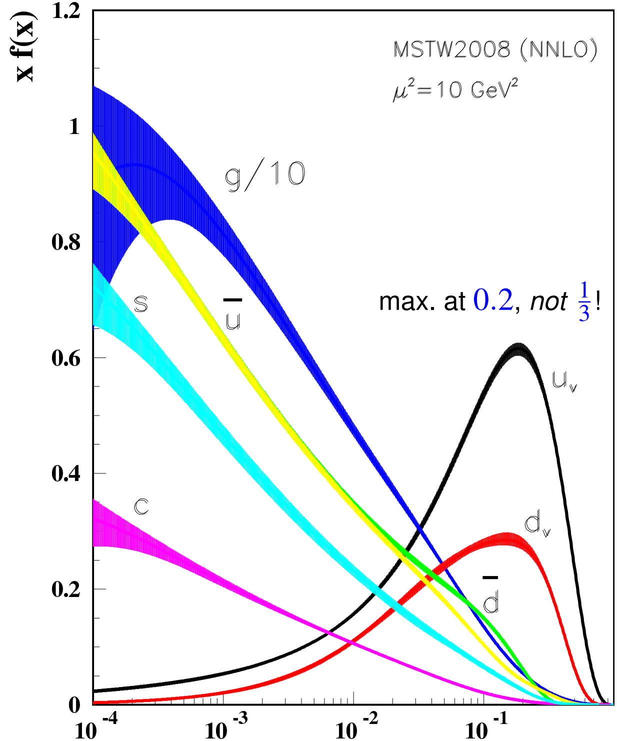
(i) Perturbative QCD in Parton Distribution Functions

Reminder of II.4: PDFs in DIS limit $Q^2 \to \infty$ depend only on Bjorken-$x = \dfrac{-q^2}{2p \cdot q} \in [0;1]$.

PDFs $q(x)$ smeared by interactions:
Especially the sea-quark distributions
depend on details of QCD!

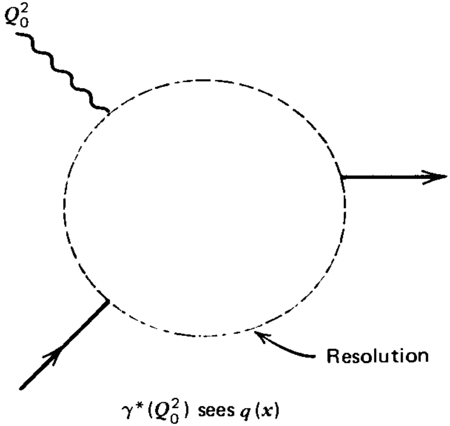




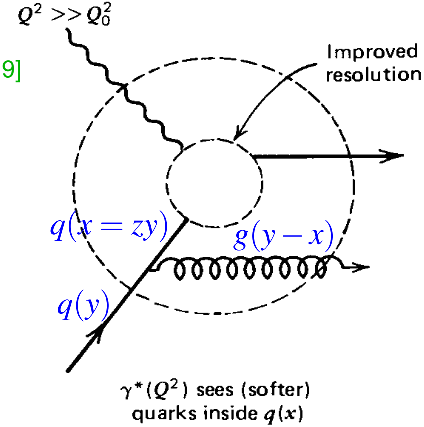


[PDG 2012 18.4]

Quark-Gluon Interactions Introduce Q^2-Dependence: $q(x,Q^2)$

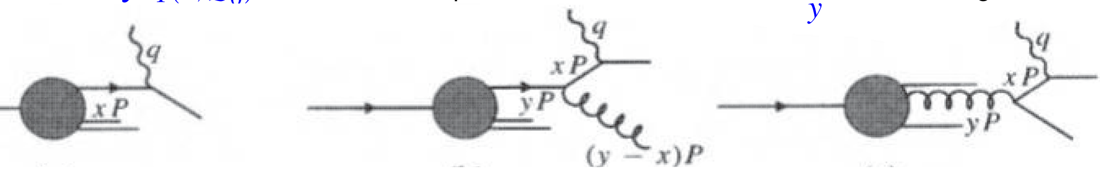


[HM 10.9]



Probability to find quark with mom. fraction $x=\xi$: $q(x,Q_0^2)$

Quark with fraction y emits gluon with mom. fraction $y-x$, so now quark carries x, or fraction $z=\frac{x}{y}<1$ of its original mom.



[Per 6.13]

Resolution increase can also create new quark with fraction x from gluon with fraction y:

Uncovers previously hidden momentum fraction, now seen by photon: extract $g(x)$!

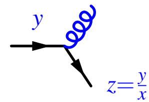
Change of Resolution: DGLAP-WW integro-differential Evolution Equations use QCD Interactions.

[Dokshitzer/Gribov/Lipatov 1972-5; Altarelli-Parisi 1977; Weizsäcker/Williams 1934 for QED]

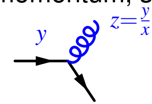
QCD Splitting Functions and DGLAP-WW

$P_{B\leftarrow A}(z)$: (prop. to) probability that parton A emits parton B with fraction z of A's momentum, seen by γ.

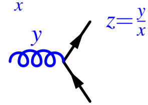
Bremsstrahlung: $P_{q\leftarrow q}(z) = P_{g\leftarrow q}(1-z) = \frac{4}{3}\,\frac{1+z^2}{1-z}$



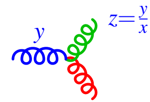
or



Gluon Annihilation: $P_{q\leftarrow g}(z) = P_{q\leftarrow g}(1-z) = \frac{1}{2}[z^2 + (1-z)^2]$



Gluon Scattering/Bremsstrahlung: $P_{g\leftarrow g}(z) = 6\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right]$



$\Longrightarrow$ **Change of Resolution leads to by DGLAP-WW Evolution Equations:**

$$\frac{\partial}{\partial \ln Q^2}\begin{pmatrix} q_i(x,Q^2) \\ g(x,Q^2) \end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi}\int\limits_x^1 \frac{\mathrm{d}y}{y}\begin{pmatrix} P_{q\leftarrow q}\left(\frac{x}{y}\right) & P_{q\leftarrow g}\left(\frac{x}{y}\right) \\ P_{g\leftarrow q}\left(\frac{x}{y}\right) & P_{g\leftarrow g}\left(\frac{x}{y}\right) \end{pmatrix}\begin{pmatrix} q_i(y,Q^2) \\ g(y,Q^2) \end{pmatrix}$$

Coupled integro-differential equations at LO in $\alpha_s < 1$.

Need *initial condition*: Complete set of PDFs at one value of Q^2. Rest prediction.

Test running of $\alpha_s(Q^2)$ *and* QCD Splitting Functions (colour factors, interactions).

Changes in $g(x,Q^2)$ ricochet into/ties together all quark flavours. $\Longrightarrow$ Find $g(x,Q^2)$.

Splitting functions get large as $z \to 0$ $\Longrightarrow$ Test with sea-quarks ($x \to 0$)!

[Dokshitzer/Gribov/Lipatov 1972-5; Altarelli-Parisi 1977; Weizsäcker/Williams 1934 for QED]

Extension to α_s^2 includes $gggg$ interaction.

Parton Distribution Functions: Scaling Violations by QCD

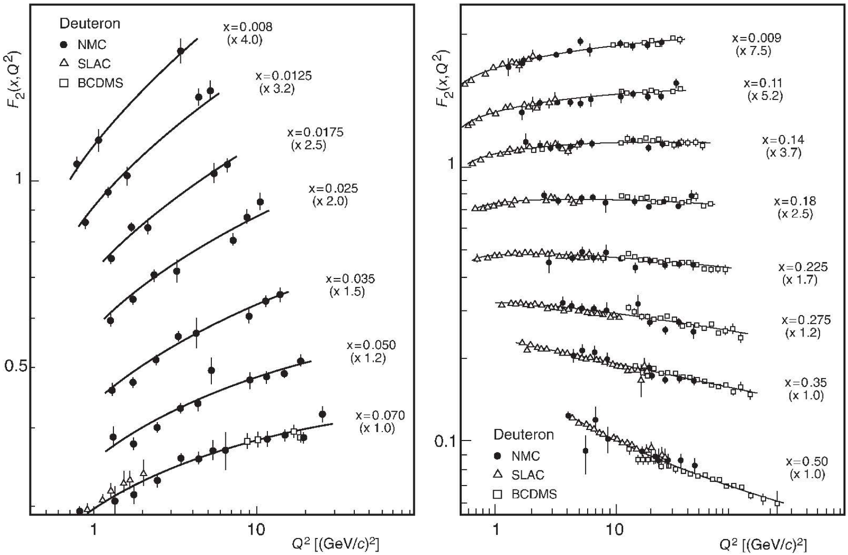


Fig. 8.5. Structure function F_2 of the deuteron as a function of Q^2 at different values of x on a logarithmic scale. The results shown are from muon scattering at CERN (NMC and BCDMS collaboration) [Am92a, Be90b] and from electron scattering at SLAC [Wh92]. For clarity, the data at the various values of x are multiplied by constant factors. The solid line is a QCG fit, taking into account the theoretically predicted scaling violation. The gluon distribution and the strong coupling constant are free parameters here.

[PRSZR]

Results:

– Excellent agreement with QCD.

– Extract gluon-PDFs & $\alpha_s(Q^2)$.

– $x > 0.2$ (valence dominate):
$F_2(x = \text{const.}, Q^2) \searrow$ as $Q^2 \nearrow$:
Gluon radiation sucks momentum from valence quarks, gives to sea.

– $x < 0.2$ (sea & glue dominate):
$F_2(x = \text{const.}, Q^2) \nearrow$ as $Q^2 \nearrow$.

– Lattice QCD starts to solve for PDFs.
$\Longrightarrow$ Provides initial condition & evolution into confinement region $\alpha_s \geq 1$ beyond perturbation theory.

Next: 3. Lattice QCD

Familiarise yourself with: [(Path Integral: Ryd 5; Sakurai: Modern QM 2.5); CL 10.5; PDG 18; Wagner [arXiv:1310.1760 [hep-lat]]; Alexandru, Lee, Freeman, Lujan, Guo;. . .]