PHYS 6610: Graduate Nuclear and Particle Physics I

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III. Descriptions

3. Lattice QCD

Or: Using Large Computers for Fun

References: [(Path Integral: Ryd 5; Sakurai: Modern QM 2.5); CL 10.5; PDG 17; Wagner [arXiv:1310.1760 [hep-lat]]; Alexandru, Lee, Lujan,. . .]

(a) Motivation of the Path Integral

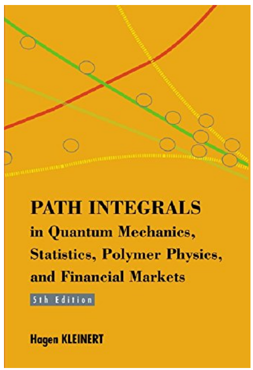
[Ryd 5; Sakurai: Mod. QM 2.5]

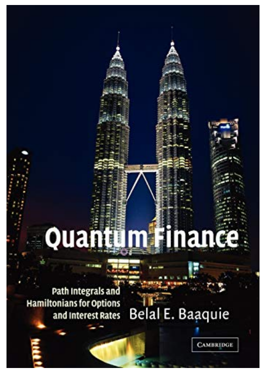
Historic Note

Thirty-one years ago, Dick Feynman told me about his 'sum over histories' version of quantum mechanics. 'The electron does anything it likes', he said. 'It goes in any direction at any speed, forward or backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function.' I said to him, 'You're crazy'. But he wasn't.

[Ryd Chap. 5]

F.J. Dyson‡





Kleinert/Duru solved the H atom using Feynman's Path Integral formalism in 1979 (!).

(b) Path Integrals on a Computer

Sample action of very many paths: Approximate Path Integral by Monte Carlo!

(1) Euclidean Time (Analytic Continuation/Wick Rotation)

Phase factor $\mathrm{e}^{\mathrm{i}S}$: rapid oscillations, numerically bad. $\Longrightarrow$ use **Euclidean Time**

$$\mathrm{i}S[\Phi] = +\frac{\mathrm{i}}{2}\int \mathrm{d}t\,\mathrm{d}^3r\left[\left(\frac{\partial\Phi(t,\vec{r})}{\partial t}\right)^2 - V(t,\vec{r})\right]$$

$\boxed{\text{Euclideanise } \mathrm{i}t \to \tau_E}$

$$\to -\frac{1}{2}\int \mathrm{d}\tau_E\,\mathrm{d}^3r\left[\left(\frac{\partial\Phi(\tau_E)}{\partial\tau_E}\right)^2 + V(\tau_E,\vec{r})\right] =: -S_E[\Phi]$$

$\mathrm{e}^{\mathrm{i}\frac{S}{\hbar}} \to \mathrm{e}^{-\frac{S_E}{\hbar}}$: $L = T - V$ in $3+1$ spacetime dim.

$\Longrightarrow$ QM Partition Function $\mathrm{e}^{-\beta H}$ with $H = T + V$ in 4 spatial dim. at temperature $\beta = \frac{\mathrm{i}}{\hbar}(t_f - t_i)$.

Disadvantages:

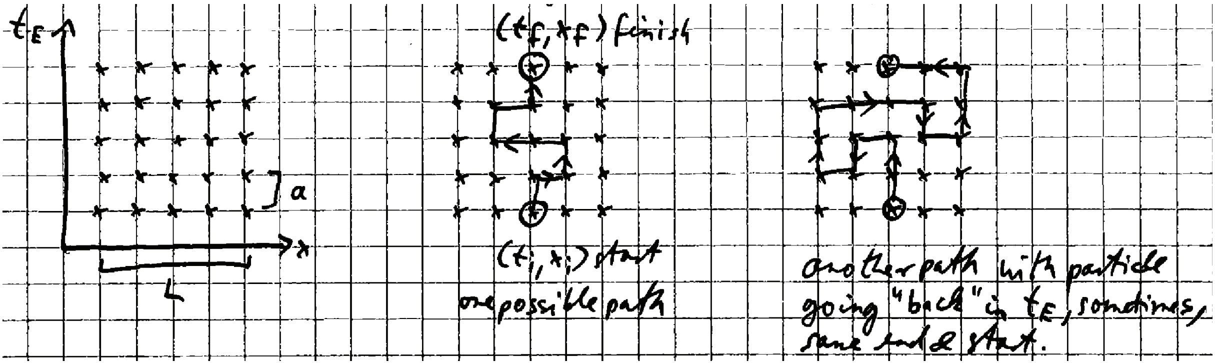
–τ_E is *not* a time – just a parameter.

– No guarantee that analytic continuation to real time t describes *same* QM system – but so far, all explored cases do (proofs perturbative).

Advantages: Lattice QCD in $3+1$ dimensions is Statistical Mechanics in 4 dim.: Same techniques!

(2) Discretisation

Simplest: 4-dim. lattice with spacing a, N points, length $L = Na$ in each direction.



Computer probes large but finite number if paths: Particle "hops" from point to point, calculate action, weight with e^{-S_E}, sum over many configurations:

$$\mathcal{A}_E = \langle x_{E,f}, \vec{r}_f | \sum_{\text{paths}} \mathrm{e}^{-S_E} | x_{E,i}, \vec{r}_i \rangle$$

cf. Ising Model of Statistical Mechanics

(3) Taking The Limits

Continuum Limit $a \to 0$

Thermodynamic/Infinite-Volume Limit $L \to \infty$ (esp. $\Delta t = Na \widehat{=} \beta = \frac{1}{\text{temperature}} \to \infty$)

Set dimension-ful **scale** $\Longrightarrow$ later…

In Practise

All of hadron comfortably inside box: $L \gg \sqrt{\langle r^2 \rangle} \approx 1\text{fm}$ – today's standard: $L > 3\text{fm}$.

Discretisation error: $0 \leftarrow a \ll$ typ. short-distance scale $\sim \dfrac{1}{2\text{GeV}} \approx 0.1\text{fm}$.

Asymptotic Freedom of QCD helps: $g \searrow 0$ for large q

$\Longrightarrow$ asymptotically non-interacting theory "shields" $a \to 0$ details.

Lattice QCD: perturbation hard $\Longrightarrow$ find intermediate match.

Today's Lattices

$N = \dfrac{L}{a} > 30$ per dimension ($d = 4$), each has spin (4-dim. spinor), colour (3×3), ≥ 3 flavours (uds).

$(N = 64)^4 \approx 10^7$ doable, $32^4 \approx 10^6$ standard.

Huge matrix describes how particles hop from one point to next (usually sparse).

(d) Free Fields on the Lattice

Points $\to$ Fields: $x(t) \to \Phi(x^\mu)$ For A Massive Real Scalar Field

Consider 1-Dimensional Case: only time direction, nothing else – generalisation straightforward.

$$\mathrm{i}S[\Phi] = +\frac{\mathrm{i}}{2}\int \mathrm{d}t \left[\left(\frac{\partial\Phi(t)}{\partial t}\right)^2 - m^2\Phi^2(t)\right]$$

Euclideanise $\mathrm{i}t \to \tau_E$

$$\to -\frac{1}{2}\int \mathrm{d}\tau_E \left[\left(\frac{\partial\Phi(\tau_E)}{\partial\tau_E}\right)^2 + m^2\Phi^2(\tau_E)\right] =: -S_E[\Phi]$$

Rewrite as 2nd derivative

$$\to -\frac{1}{2}\int \mathrm{d}\tau_E\, \Phi(\tau_E)\left[-\frac{\partial^2}{\partial\tau_E^2} + m^2\right]\Phi(\tau_E)$$

Discretise á la Runge-Kutta RK2

$$\to -\frac{a}{2}\sum_{\text{lattice sites } n}\left[\Phi_n\,\frac{-1}{a^2}\left(\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}\right) + m^2\Phi_n^2\right]$$

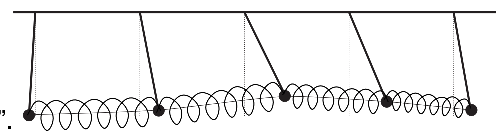
Convert to matrix on vector $\vec{\Phi} = \begin{pmatrix}\Phi_1\\ \vdots \\ \Phi_N\end{pmatrix}$

$$\to\propto \vec{\Phi}^{\mathrm{T}} \begin{pmatrix} -\frac{2}{a^2}+m^2 & \frac{1}{a^2} & 0 & 0 & \dots \\ \frac{1}{a^2} & -\frac{2}{a^2}+m^2 & \frac{1}{a^2} & 0 & \dots \\ 0 & \frac{1}{a^2} & -\frac{2}{a^2}+m^2 & \frac{1}{a^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \vec{\Phi}$$

This is a **Linear Chain of Coupled Harmonic Oscillators**:

Dislocation Φ_n at point n by **spring with constant** $\propto \dfrac{1}{a^2}$,

nearest-neighbour interactions, m provides additional “drag”.



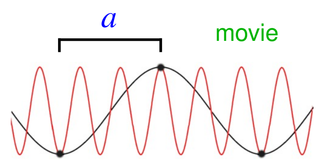
Momentum Restriction: Brillouin Zone and Miniumum Momentum

$$S_E[\Phi] = \frac{a}{2} \sum_{\text{lattice sites } n} \left[\Phi_n \frac{-1}{a^2} (\Phi_{n+1} - 2\Phi_n + \Phi_{n-1}) + m^2 \Phi_n^2 \right]$$

Solve by **Discrete Fourier Transform** $\Phi_n = \int \frac{\mathrm{d}k}{2\pi} \, \mathrm{e}^{\mathrm{i}kna} \, \Phi(k)$ at momentum k. $\rightarrow$ HW

Result: Correct continuum limit for relativistic E-p relation: $\frac{1}{\text{propagator}} \xrightarrow{a \to 0} m^2 + k^2 + \mathcal{O}(a^2)$.

Resolution a cannot resolve high momenta/high-frequency oscillations.



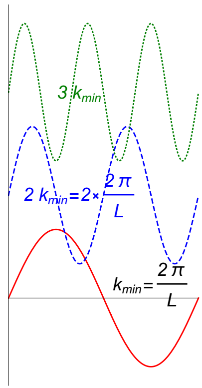
black: $k < \frac{\pi}{a}$; red: $k > \frac{\pi}{a}$

$\Longrightarrow$ **Useful momenta must be inside *Brillouin Zone*** $-\frac{\pi}{a} \leq k \leq \frac{\pi}{a}$.

In *finite* lattice volume, there is also a **smallest nonzero momentum**.

Example hypercube with Periodic Boundary Condition $\Phi_n = \Phi_{n+N}$:

$$k_{\min} = \pm \frac{2\pi}{L}$$



$\Longrightarrow$ **Momenta are discretised:** $k = 0, \pm\frac{2\pi}{L}, \pm\frac{4\pi}{L}, \dots, \pm\frac{\pi}{a}$

(e) QCD on the Lattice (pretend no matrices: simpler – results hold for matrices)

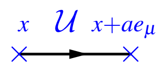
$$\mathcal{L}_{QCD} = \overline{\Psi}\, \mathrm{i}\gamma^\mu \left[\partial_\mu - \mathrm{i}g\, A_\mu\right] \Psi - m\, \overline{\Psi}\Psi - \frac{1}{2}\,\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}]$$

Discretisation: Quarks Ψ on lattice sites, derivative connects adjacent sites.

A_μ has direction, $A_\mu \xrightarrow{\text{G.T.}} U[A_\mu + \frac{\mathrm{i}}{g}\,\partial_\mu]U^\dagger \implies$ How to keep gauge invariance for $a \neq 0$?

Idea: gauge-covariant derivative $D_\mu := \partial_\mu - \mathrm{i}g\, A_\mu \implies A_\mu$ **links** adjacent lattice points in μ-direction.

Link Variable $\mathcal{U}[x + ae_\mu \leftarrow x] \equiv \exp \mathrm{i}g \int\limits_0^a \mathrm{d}\tau\, A_\mu(x + \tau e_\mu)$



e_μ: unit vector in μ-direction

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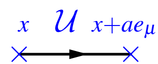

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Gauge Transformation $\Psi(x) \to \mathrm{e}^{\mathrm{i}g\alpha(x)}\Psi(x) \equiv U(x)\;\Psi(x)$:

again: no matrices to simplify!

$$\mathcal{U}[x + ae_\mu \leftarrow x] \xrightarrow{\text{G.T.}} \exp \mathrm{i}g \int\limits_0^a \mathrm{d}\tau\;\left[A_\mu(x + \tau e_\mu) + \partial_\mu \alpha(x + \tau e_\mu)\right]$$

integral of derivative

$$= \exp[\mathrm{i}g\;\alpha(x + ae_\mu) + \left[\mathrm{i}g \int\limits_0^a \mathrm{d}\tau\, A_\mu(x + \tau e_\mu)\right] - \mathrm{i}g\;\alpha(x)]$$

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split exponents

$$= \mathrm{e}^{\mathrm{i}g\, \alpha(x + ae_\mu)} \left[\exp \mathrm{i}g \int\limits_0^a \mathrm{d}\tau\, A_\mu(x + \tau e_\mu)\right] \mathrm{e}^{\mathrm{i}g\, \alpha(x)}$$

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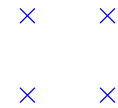

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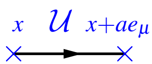

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use definitions $\quad = U(x + ae_\mu)\, \mathcal{U}[x + ae_\mu \leftarrow x]\; U^\dagger(x)$

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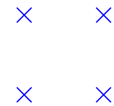

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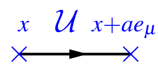

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$\xrightarrow{ga\to 0} \overline{\Psi}(x+ae_\mu)\left[1 + \mathrm{i}g\,a\,A_\mu(x+\frac{a}{2}e_\mu) + \mathcal{O}(ga)^2\right]\Psi(x)$ midpoint approximation for integral

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midpoint approximation for integral

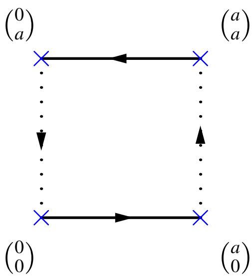
$$= \underbrace{\overline{\Psi}(x+ae_\mu)\Psi(x)}_{\to\,\overline{\Psi}\partial_\mu\Psi} + \underbrace{\overline{\Psi}(x+ae_\mu)\,\mathrm{i}g\,a\,A_\mu(x+\frac{a}{2}e_\mu)\,\Psi(x)}_{\to\ \text{interaction by link between sites}}$$

and gives correct $a \to 0$ limit.

$\implies$ Use **link variables** $\mathcal{U} \in SU(N)$: gauge invariance for all $a \neq 0$, no direct reference to A_μ!

Plaquette:

4 adjacent sites
with their links



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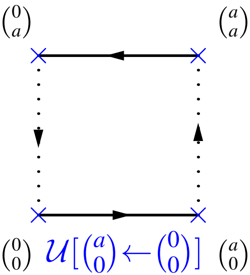

Compare two parallel links 1 lattice spacing apart in (xy)-plane:

$$\mathcal{U}[\begin{pmatrix}a\\0\end{pmatrix}\leftarrow\begin{pmatrix}0\\0\end{pmatrix}] = \exp \mathrm{i}g\int\limits_0^a \mathrm{d}\tau\, A_x\begin{pmatrix}\tau\\0\end{pmatrix} \xrightarrow{a\to 0} 1+\mathrm{i}g\, a\, A_x\begin{pmatrix}\frac{a}{2}\\0\end{pmatrix}+\mathcal{O}(ga)^2$$

Gluon Action: Field Strength Tensor on the Lattice (again: non-matrix case)

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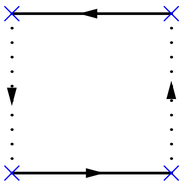
$\binom{0}{a}$ $\mathcal{U}[\binom{0}{a} \leftarrow \binom{a}{a}]$ $\binom{a}{a}$



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$$\mathcal{U}[\begin{pmatrix} 0 \\ a \end{pmatrix} \leftarrow \begin{pmatrix} a \\ a \end{pmatrix}] = \mathcal{U}^\dagger[\begin{pmatrix} a \\ a \end{pmatrix} \leftarrow \begin{pmatrix} 0 \\ a \end{pmatrix}]$$

reverse integration direction

$$= \exp - \mathrm{i}g \int\limits_0^a \mathrm{d}\tau\, A_x \begin{pmatrix} \tau \\ a \end{pmatrix} \xrightarrow{a \to 0} 1 - \mathrm{i}g\, a\, A_x \begin{pmatrix} \frac{a}{2} \\ a \end{pmatrix} + \mathcal{O}(ga)^2$$

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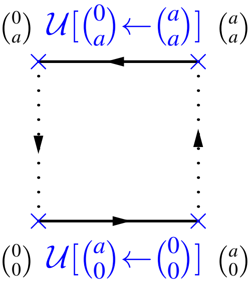
Plaquette:

4 adjacent sites with their links



Compare two parallel links 1 lattice spacing apart in (xy)-plane:

$$\mathcal{U}[\begin{pmatrix}a\\0\end{pmatrix}\leftarrow\begin{pmatrix}0\\0\end{pmatrix}] = \exp \mathrm{i}g\int\limits_0^a \mathrm{d}\tau\, A_x\begin{pmatrix}\tau\\0\end{pmatrix} \xrightarrow{a\to 0} 1+\mathrm{i}g\,a\,A_x\begin{pmatrix}\frac{a}{2}\\0\end{pmatrix}+\mathcal{O}(ga)^2$$

$$\mathcal{U}[\begin{pmatrix}0\\a\end{pmatrix}\leftarrow\begin{pmatrix}a\\a\end{pmatrix}] = \mathcal{U}^\dagger[\begin{pmatrix}a\\a\end{pmatrix}\leftarrow\begin{pmatrix}0\\a\end{pmatrix}]$$

reverse integration direction

$$= \exp -\mathrm{i}g\int\limits_0^a \mathrm{d}\tau\, A_x\begin{pmatrix}\tau\\a\end{pmatrix} \xrightarrow{a\to 0} 1-\mathrm{i}g\,a\,A_x\begin{pmatrix}\frac{a}{2}\\a\end{pmatrix}+\mathcal{O}(ga)^2$$

Combine: $\mathcal{U}[\begin{pmatrix}0\\a\end{pmatrix}\leftarrow\begin{pmatrix}a\\a\end{pmatrix}]\,\mathcal{U}[\begin{pmatrix}a\\0\end{pmatrix}\leftarrow\begin{pmatrix}0\\0\end{pmatrix}] = 1+\mathrm{i}g\,a\underbrace{\left[A_x\begin{pmatrix}\frac{a}{2}\\0\end{pmatrix}-A_x\begin{pmatrix}\frac{a}{2}\\a\end{pmatrix}\right]}_{\xrightarrow{a\to 0} -a\,\partial_y A_x\begin{pmatrix}a/2\\a/2\end{pmatrix}}+\mathcal{O}(ga)^2$

Looks like part of Field Strength Tensor $F_{xy}=\partial_x A_y-\partial_y A_x$

Gluon Action: Field Strength Tensor on the Lattice (again: non-matrix case)

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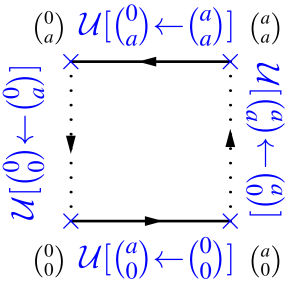

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Looks like part of Field Strength Tensor $F_{xy}=\partial_x A_y-\partial_y A_x$

Combine with the other 2 parallel links for gauge-invariant expression (holds also with matrices):

Plaquette Variable

$$\mathcal{U}_\Box := \mathcal{U}[\begin{pmatrix}0\\0\end{pmatrix}\leftarrow\begin{pmatrix}0\\a\end{pmatrix}]\,\mathcal{U}[\begin{pmatrix}0\\a\end{pmatrix}\leftarrow\begin{pmatrix}a\\a\end{pmatrix}]\,\mathcal{U}[\begin{pmatrix}a\\a\end{pmatrix}\leftarrow\begin{pmatrix}a\\0\end{pmatrix}]\,\mathcal{U}[\begin{pmatrix}a\\0\end{pmatrix}\leftarrow\begin{pmatrix}0\\0\end{pmatrix}]$$

$$= 1+\mathrm{i}g\, a^2\underbrace{(\partial_x A_y-\partial_y A_x)}_{=F_{xy}}+\mathcal{O}(ga)^2$$

like Field Strength Tensor

QCD Action on the Lattice

In QCD, $\mathcal{U}[x+ae_\mu \leftarrow x]$, $\mathcal{U}_\Box \in SU(3)$, $F_{xy} = \partial_x A_y - \partial_y A_x - \mathrm{i}g\,[A_x, A_y]$ contains self-interactions.

But same relation between F and $\mathcal{U}_\Box$ holds: $$F_{xy} = \frac{1}{\mathrm{i}a^2}\,\frac{1}{g}\,[\mathcal{U}_\Box - 1] + \mathcal{O}(ga)^0$$

Gluon action $\displaystyle\int \mathrm{d}^4x\, \mathcal{L}_{\text{glue}} = -\frac{1}{2g^2}\sum \mathrm{tr}[\mathcal{U}_\Box]$: $\sum$um over all elementary plaquettes on lattice

$\rightarrow 0$ **for** $g \rightarrow \infty$**: strong-coupling limit is easy on lattice!** (hard in perturbation: complementing)

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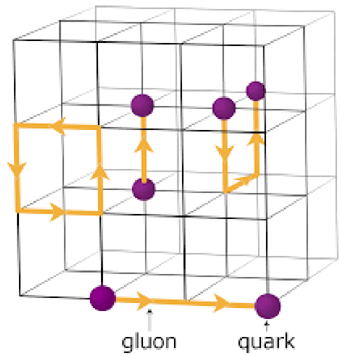
$\Longrightarrow$ **Insert QCD action on lattice into Euclidean Path Integral:**

$$\mathcal{Z} = \int \underbrace{[\mathcal{D}\overline{\Psi}]\;[\mathcal{D}\Psi]}_{\text{all quarks on sites}}\;\underbrace{[\mathcal{D}\mathcal{U}]}_{\substack{\text{all link}\\ \text{variables}\\ \text{as } SU(3)\\ \text{matrices}}}\;\exp - \sum_{\text{sites,links}} \mathcal{L}[\overline{\Psi},\Psi;\mathcal{U},\mathcal{U}_\Box[\mathcal{U}]]$$



A_μ replaced by matrices $\mathcal{U} \in SU(3)$. Plaquette variables $\mathcal{U}_\Box[\mathcal{U}]$ are functions of link variables $\mathcal{U}$.

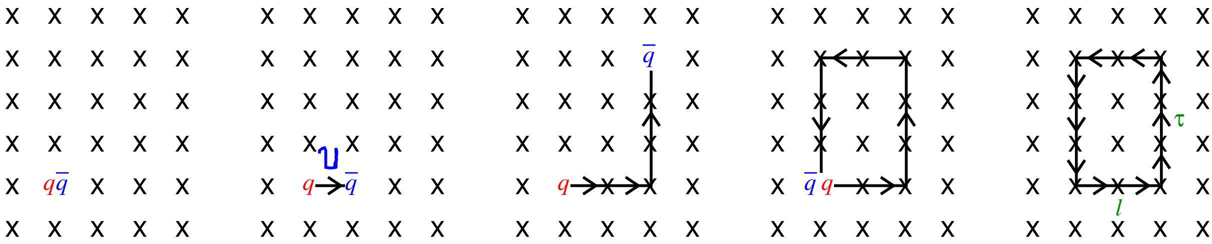
To solve, go to Monte Carlo & play dice for random links and sites on 4-dimensional hypercube:

$\mathcal{Z}$ is partition function of Statistical Mechanics in 4-dimensional space (no time).

(f) Static Quarks, Strong Coupling, Confinement

Wilson 1974

Infinitely heavy quarks $\Longrightarrow$ action $m_q\,\bar{q}q \to \infty$ $\Longrightarrow$ quarks classical $\Longrightarrow$ quarks not in PI.



$q\bar{q}$ pair on site, drag $\bar{q}$ away $\Longrightarrow$ gauge inv. $\bar{q}(y)\mathcal{U}(y\leftarrow x)q(x)$, close loop $\mathcal{C}(\tau,l)$, annihilate $q\bar{q}$.

Wilson Loop $W[\mathcal{C}(\tau,l)] = \prod\limits_{\text{links } i \text{ along } \mathcal{C}} \mathcal{U}_i$ (index $i=1,\dots$ labels links between sites of lattice)

is extension of plaquette $\mathcal{U}_\square$ to rectangle $\mathcal{C}(\tau,l)$ with sides (τ,l) in lattice units;

represents static $q\bar{q}$ pair at distance l existing for (Euclidean) "time" τ.

Expectation value for energy of static $q\bar{q}$ pair (static loop $\mathcal{C}$) (kin.energy $T=0$) only from gluon action:

$$\langle q\bar{q}(\tau,l)|\mathrm{e}^{-Ht_E}|q\bar{q}(0,l)\rangle = \int \prod_{\text{all links } i} \mathrm{d}\mathcal{U}_i \left[\prod_{\substack{\text{links } j\\ \text{along } \mathcal{C}}} \mathcal{U}_j\right] \mathrm{e}^{-\left[-\frac{1}{2g^2}\sum\limits_{\text{plaquettes}} \mathrm{tr}\,\mathcal{U}_\square\right]} \propto \mathrm{e}^{-(T+V)\tau} \overset{T=0}{\equiv} \mathrm{e}^{-V(l)\tau}$$

Calculate the Path Integral in the Strong Coupling Limit of $U(1)$

$$\langle q\bar{q}(\tau,l)|\mathrm{e}^{-Ht_E}|q\bar{q}(0,l)\rangle = \int \prod_{\text{all links } i} \mathrm{d}\mathcal{U}_i \Big[\prod_{\substack{\text{links } j\\ \text{along } \mathcal{C}}} \mathcal{U}_j\Big] \mathrm{e}^{+\frac{1}{2g^2}\sum\limits_{\text{plaquettes}} \mathrm{tr}\,\mathcal{U}_\square} \propto \mathrm{e}^{-(T+V)x} \stackrel{T=0}{=} \mathrm{e}^{-V(l)\tau}$$

$U(1)$ gauge group: each $\mathcal{U}_i = \mathrm{e}^{i\alpha_i}$ on link i is number on unit circle, with some angle $\alpha_i \in [0;2\pi[$.

Integration over angles: $\int\limits_0^{2\pi} \mathrm{d}\alpha\ \mathrm{e}^{in\alpha} = \delta_{n0}$ for $n \in \mathbb{Z}$ nonzero only if no phase (average on circle is 0).

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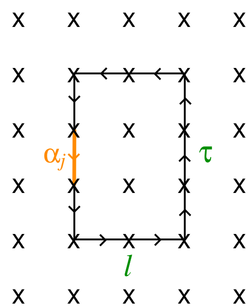
Strong Coupling Limit $g \to \infty$: expand $\exp$ of gluon action:



$$\mathcal{O}(g^0): \int \prod_{\substack{\text{all links}\\ i \text{ on lattice}}} \mathrm{d}\alpha_i \Big[\prod_{\substack{\text{only links } j\\ \text{along } \mathcal{C}}} \mathrm{e}^{\mathrm{i}\alpha_j}\Big] = 0$$

since for lattice link $i =$ link j of Wilson loop: $\int\limits_0^{2\pi} \mathrm{d}\alpha_i\ \mathrm{e}^{\mathrm{i}\alpha_{i=j}} = 0$.

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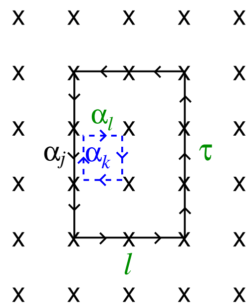
$\mathcal{O}(g^{-2})$: $\displaystyle\frac{1}{2g^2}\int \prod_{\substack{\text{all links}\\ i \text{ on lattice}}} \mathrm{d}\alpha_i \left[\prod_{\substack{\text{only links } j\\ \text{along } \mathcal{C}}} \mathrm{e}^{\mathrm{i}\alpha_j}\right] \sum_{\substack{\text{all links } klmn \text{ of}\\ \text{all lattice plaquettes}}} \mathrm{e}^{\mathrm{i}(\alpha_k+\alpha_l+\alpha_m+\alpha_n)}$



Match one link k of plaquette to link j in $\mathcal{C}$: $\alpha_j = -\alpha_k$ equal and opposite angles.

$\Longrightarrow$ One integral $\neq 0$, but all others still 0.

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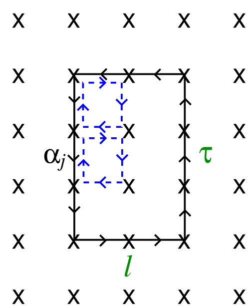

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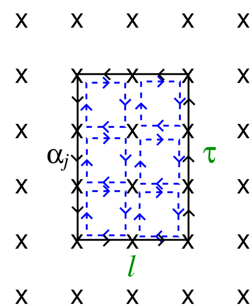

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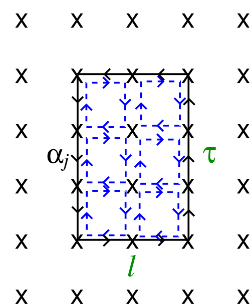

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Linear potential $V(l) = \sigma\, l$ of $q\bar{q}$ pair for strong $g \to \infty$, $\sigma \propto \ln g^2$: Flux tube/string potential!

$\Longrightarrow$ Every Gauge Theory has Confinement, in any $d \geq 2$! “Christmas”! Even in $U(1)$ (QED)??

Weak-coupling limit $g \to 0$: the more plaquettes, the more important. $\Longrightarrow$ Unsolvable.

Lattice QCD “simple” for $g \gg 1$, complicated for $g \ll 1$: complements perturbative QCD.

(g) Very Rough Outline of Lattice "Computations"

Euclidean lattice $\widehat{=}$ Partition Function in 4 Euclidean dimensions. $\Longrightarrow$ Heavily borrow from Stat. Phys.

(1) Create "Pure Glue" Ensemble: Throw dice for values of $\mathcal{U}$ at each link, weighted with action in PI.

Relaxation/Updating to equilibrate system ($\rightarrow$ Ising model).

Importance Sampling: prefer configurations with small action.

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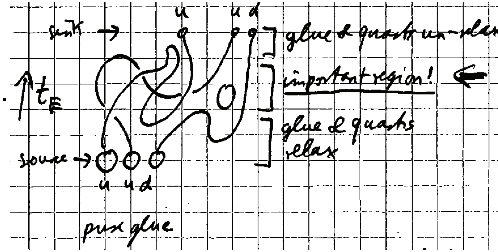

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(2) Throw in appropriate quark sources at $t_E = 0$ & T_E:

e.g. (uud) with quantum numbers of proton;
but does not need to be very accurate representation!

$$\int [\mathcal{D}\overline{\Psi}]\,[\mathcal{D}\Psi]\,[\mathcal{D}\mathcal{U}]\,\mathrm{e}^{-S_E[\overline{\Psi},\Psi;\mathcal{U}]} \propto \langle uud(T_E)|\mathrm{e}^{-Ht_E}|uud(t_E=0)\rangle$$

insert complete set of phys. states $$= \sum_n \langle uud|\mathrm{e}^{-H\,t_E}|n\rangle\langle n|uud\rangle$$

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$\Longrightarrow$ Read off energy $E_0 \equiv M$ mass of lowest state in correlation plots: **"Plateau Plots"**.

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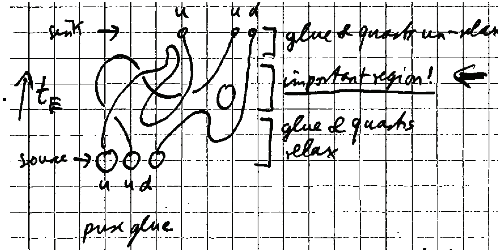

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(3) Set Scale: All quantities in units of a $\Longrightarrow$ pick few observables to fix $a, m_q, \dots$; then predict rest.

Everything (masses, energies, even input m_q) is given in units one dimension-ful quantity: lattice spacing a.

$$\lim_{\Delta t_E\to\infty}\langle \text{B-meson}(\Delta t_E)|\mathrm{e}^{-H\Delta t_E}|\text{B-meson}(t_E=0)\rangle \propto \mathrm{e}^{-\Delta t_E M_{\text{B-meson}}}$$

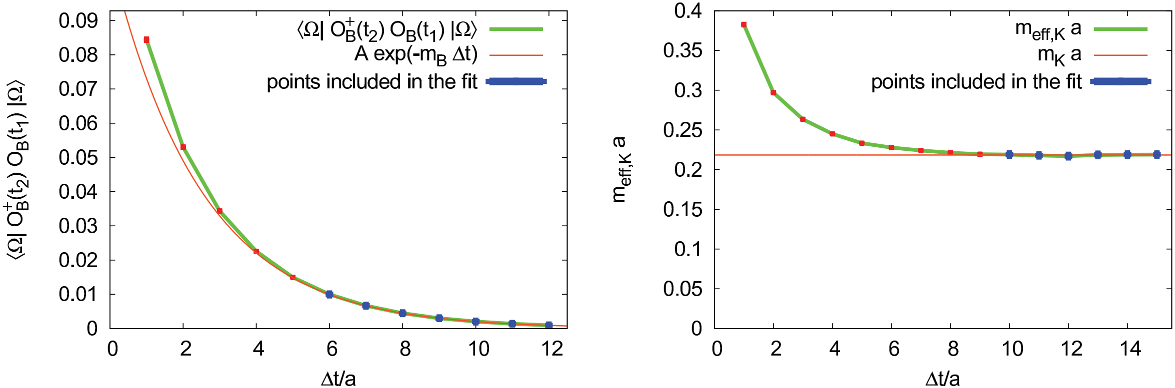
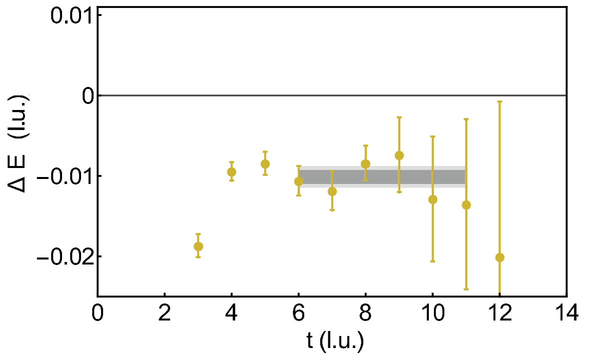


Figure 3: **left**: the temporal correlation function of a B meson creation operator as a function of the temporal separation (taken from [12]); **right**: the effective mass of the kaon as a function of the temporal separation (taken from [13]).

A More Realistic Example – And Some People's Fantasies

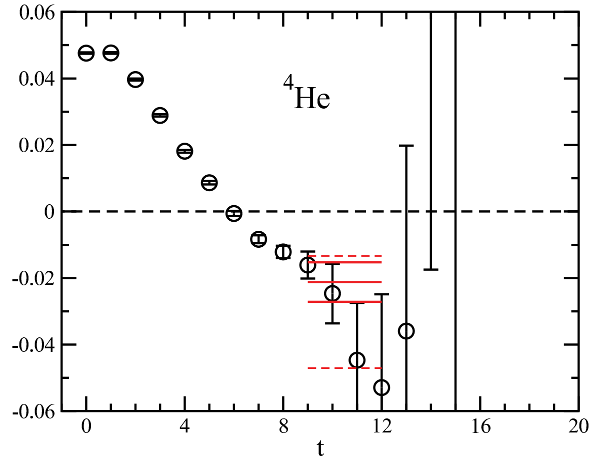


[NPLQCD [arXiv:1508.07583v1 [hep-lat]]]

Effective-mass shift $\Delta E = 2M_N - M(\text{deuteron})$ in $32^3 \times 96$ lattice, using lattice units.

Fit-error construction: At least 3 different people use different algorithms to identify plateaus independently, each providing an error estimate. Total error is statistical sum of all.

Watch out for strong correlation of points: same lattice data!



[HALQCD [arXiv:1502.04182v2 [hep-lat]]]

Eff. shift $\Delta E = 4M_N - M(^4\text{He})$ in $(4.3\text{fm})^3$ (48^4 lattice), in lattice units.

Quote: "Fit result with one standard deviation error band and total error including the systematic one is expressed by solid and dashed lines, respectively."

(h) A Few Selected Problems in Lattice QCD

QCD will be solved by Christmas. [Wilson 1974 in seminars] (Hence name "Christmas Paper".)

(1) Lattice Discretisation Breaks (Euclidean) Rotational Invariance

$\Longrightarrow$ No angular momentum conservation or partial waves, but can use discrete cubic symmetries.

Conceptually largely under control but can be numerically quite expensive; gets better as $a \to 0$.

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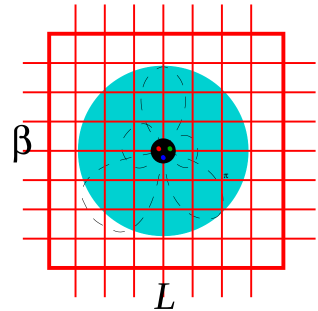
(2) Need to Fit Physics into Box i.e. “have all meson clouds inside box”



Lightest hadron: $m_\pi = 140\text{MeV} \Longrightarrow \lambda_\pi^{\text{Compton}} \sim \dfrac{\pi}{m_\pi} \simeq 4\text{fm}$,

but resolution $a \ll \dfrac{1}{\sim 1\text{GeV}} \simeq 0.1\text{fm}$.

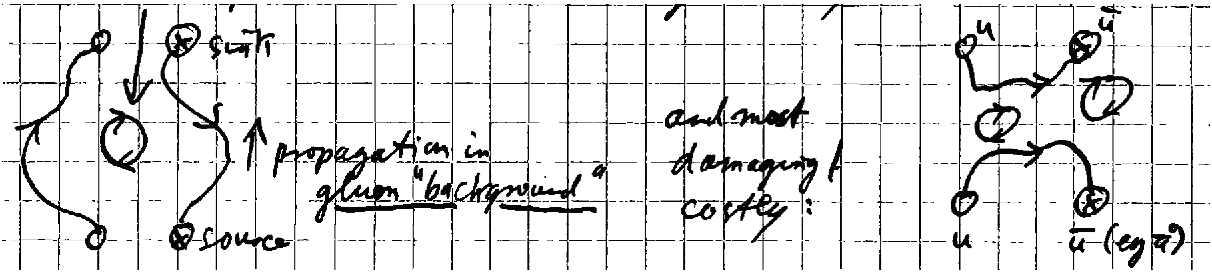
$\Longrightarrow$ Need $L \gtrsim 4\text{fm}$, $a \ll 0.1\text{fm}$, $N^4 \gg (40)^4 \approx 2.5\text{M}$ quite expensive.

Way out: Compute at larger $m_\pi \Longrightarrow$ smaller L; then extrapolate to physical m_π using model-independent, well-understood m_π-dependence of **Chiral Effective Field Theory**. $\to$ next section

(h) A Few Selected Problems in Lattice QCD

(3) Light Quarks Computationally Extremely Expensive Big problem for pion Physics from lattice.

Small energy/action penalty $e^{-S_E[m_q]}$ $\Longrightarrow$ easy to create light $q\bar{q}$ pairs in vacuum fluctuations.



Such quark-pair vacuum fluctuations are called **"disconnected"**: lines not connected to sources (cf. valence quarks). But they are still moving in gluon background: These are *not* Feynman diagrams!

Historically, such "disconnected" quark lines were just dropped: **Quenching** now only for tests.

(4) Can Only "Measure" Exponential Decays & Static Observables

$\mathrm{e}^{-H_E\, t_E} \implies$ static statistical system in 4 spatial dimensions

$\implies$ Scattering phase shifts only by induced energy-shift (Lüscher's method $\longrightarrow$ later).

Especially challenging for shallow bound states, like deuteron:

$M_{\text{deuteron}} = M_{\mathrm{p}} + M_{\mathrm{n}} - E_B \approx [2000-2]\mathrm{MeV} \implies \mathrm{e}^{-(2M_{\mathrm{N}}-E_B)\,t_E} \longleftrightarrow \mathrm{e}^{-2M_{\mathrm{N}}\,t_E}$: 0.1%-effect!

$\implies$ Need to measure tiny energy difference to two-free-nucleons!

Terrible signal-to-noise for Nuclear systems.

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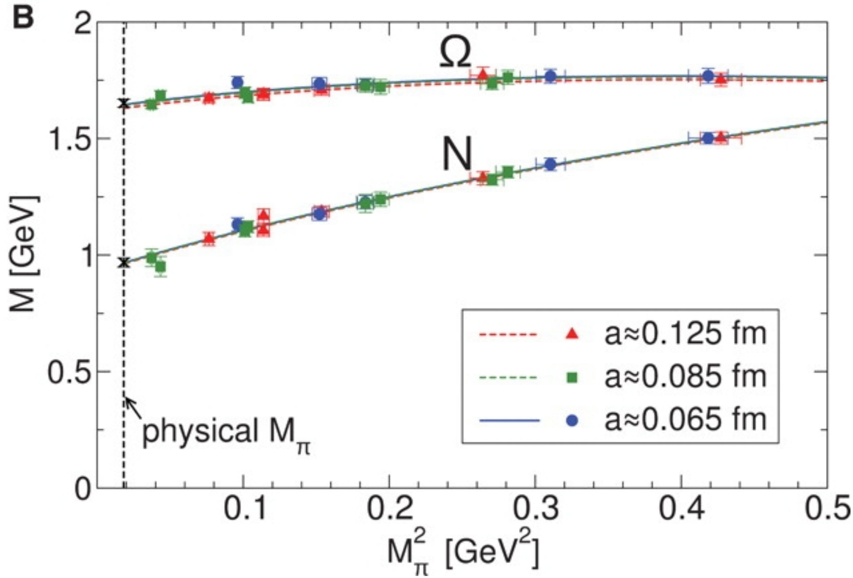
In my view, learn about Nature by combining lattice, perturbative and χEFT:
well-defined, systematically improvable descriptions of QCD, estimating theory uncertainties,
in complementing and partially overlapping energy regímes.

(i) Very Few (Even More Selected) Lattice Results

Extrapolation to Physical Masses

Use known low-energy Nuclear Physics (Chiral EFT) to cut down on computational cost.

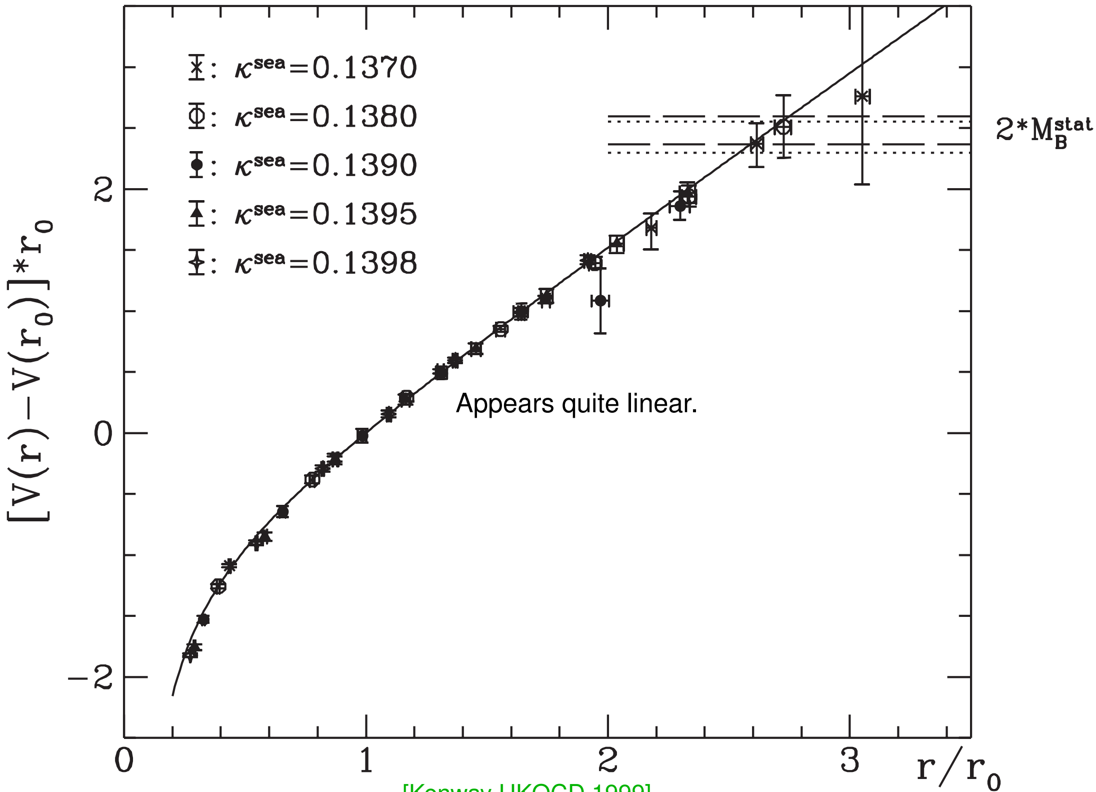
Not just a linear extrapolation!



[Duerr et al. **Science** (2008)]

Static Potential between Infinitely Heavy Quarks (Quarkonium)

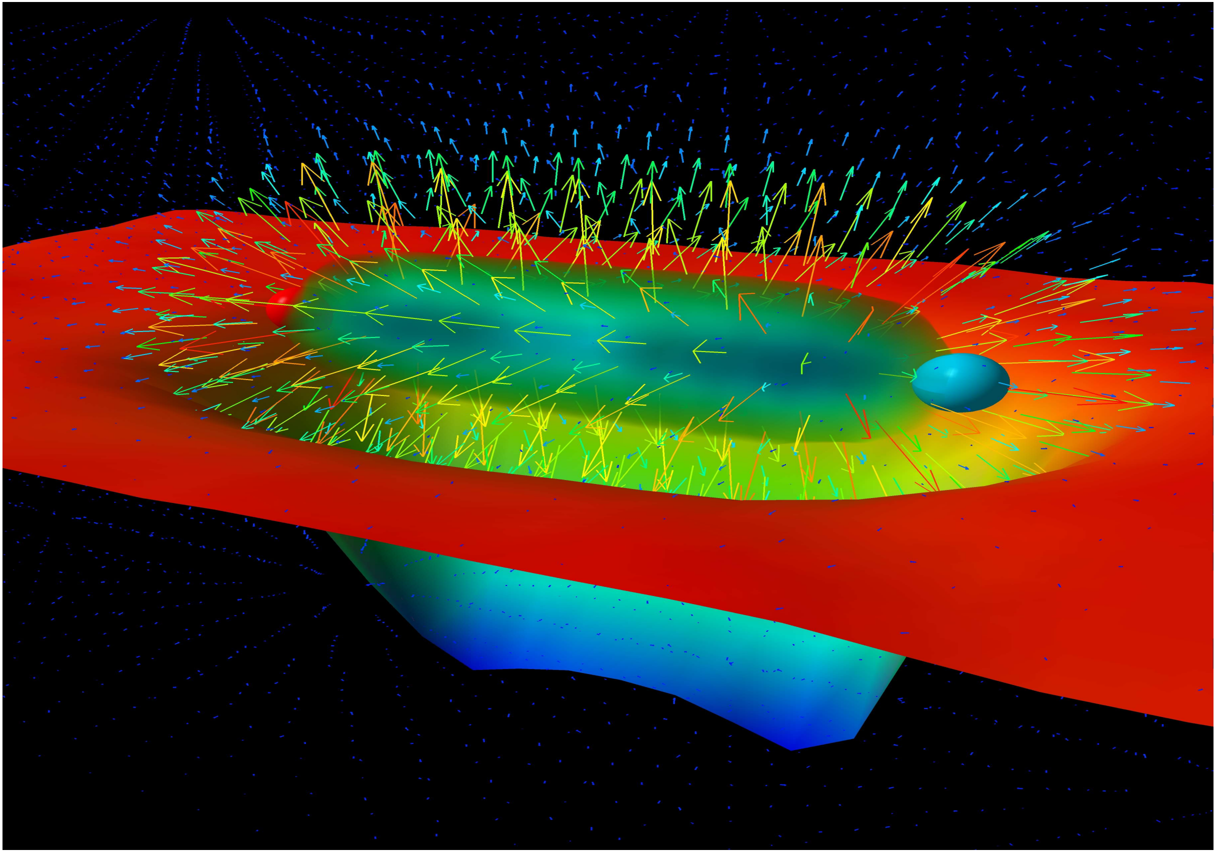
Infinitely heavy $\Longrightarrow$ no recoil $\Longrightarrow$ no retardation or colour radiation $\Longrightarrow$ **Potential makes sense.**



[Kenway UKQCD 1999]

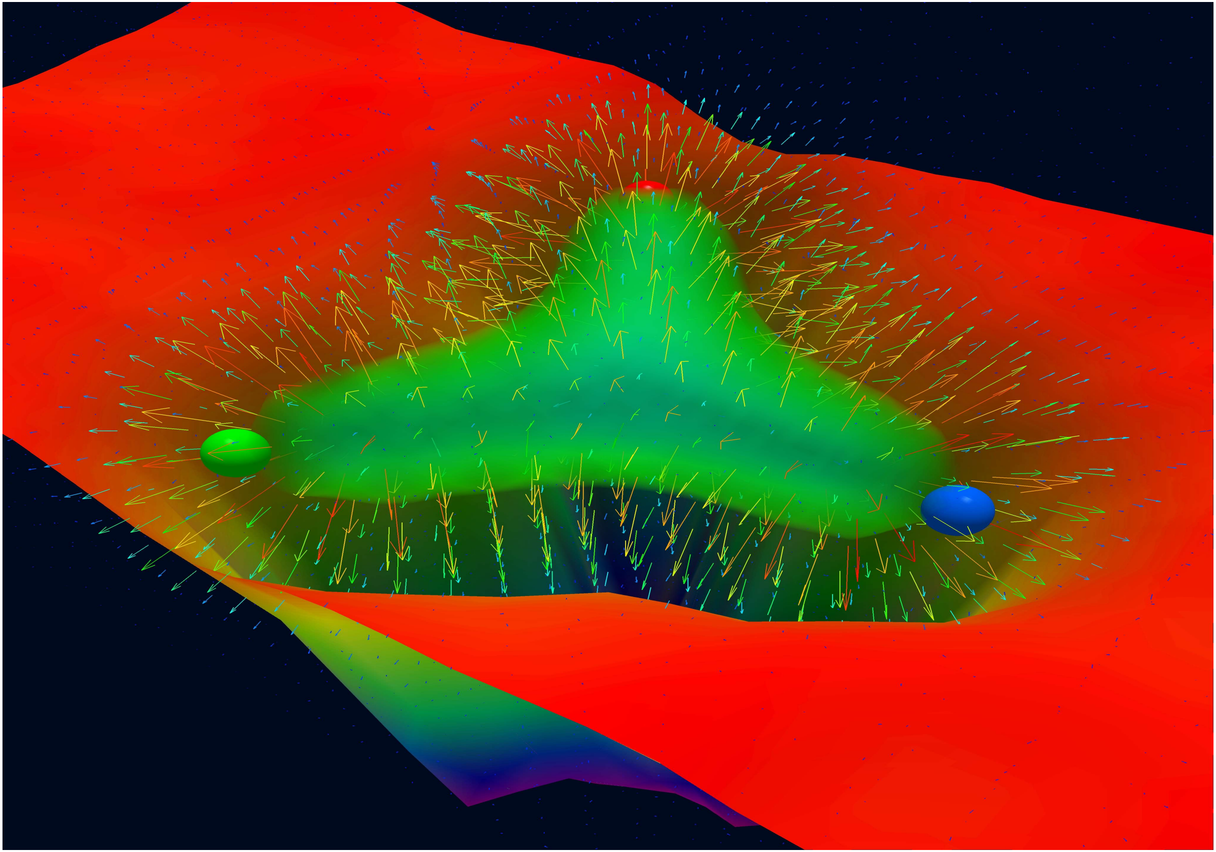
Energy Density: Flux Tube for a Heavy Meson

[Leinweber et al. 2003, click here for homepage]

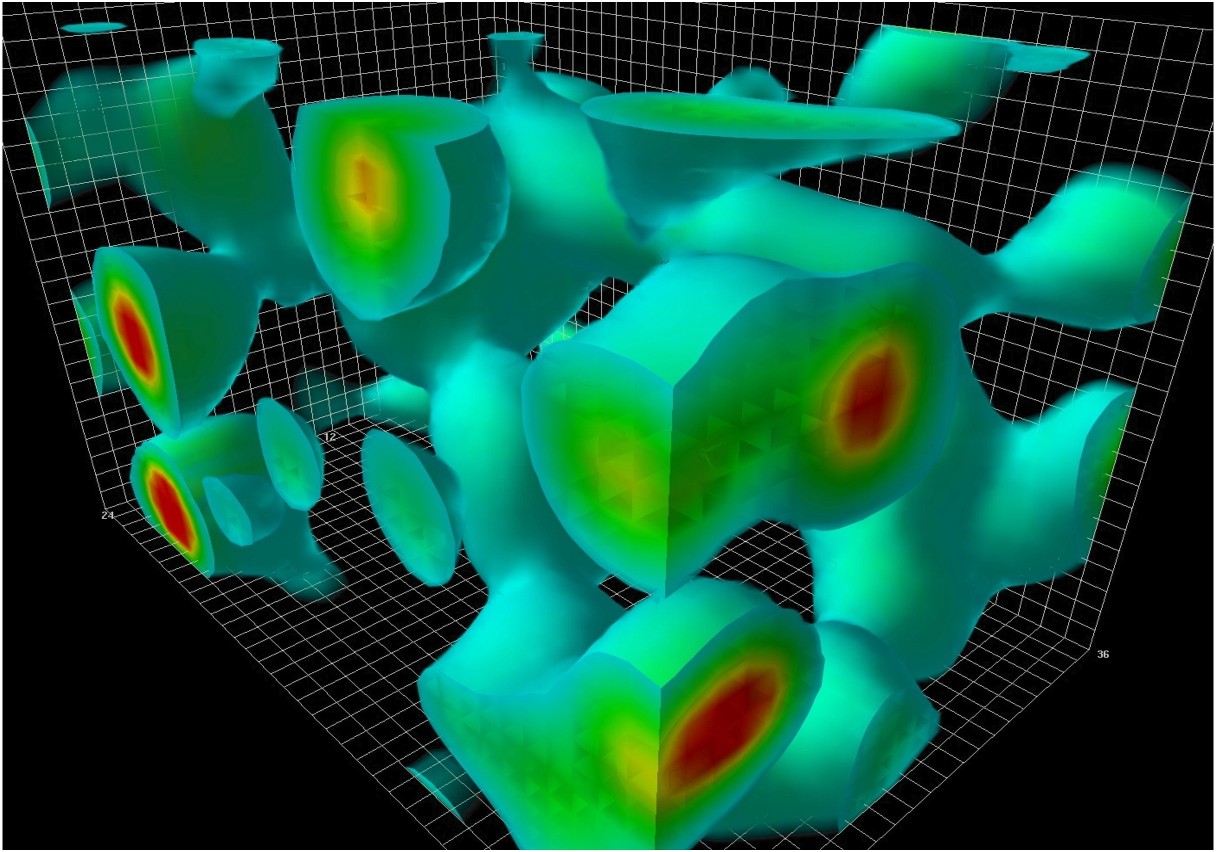


Action Density: Flux Tube for a Heavy Baryon

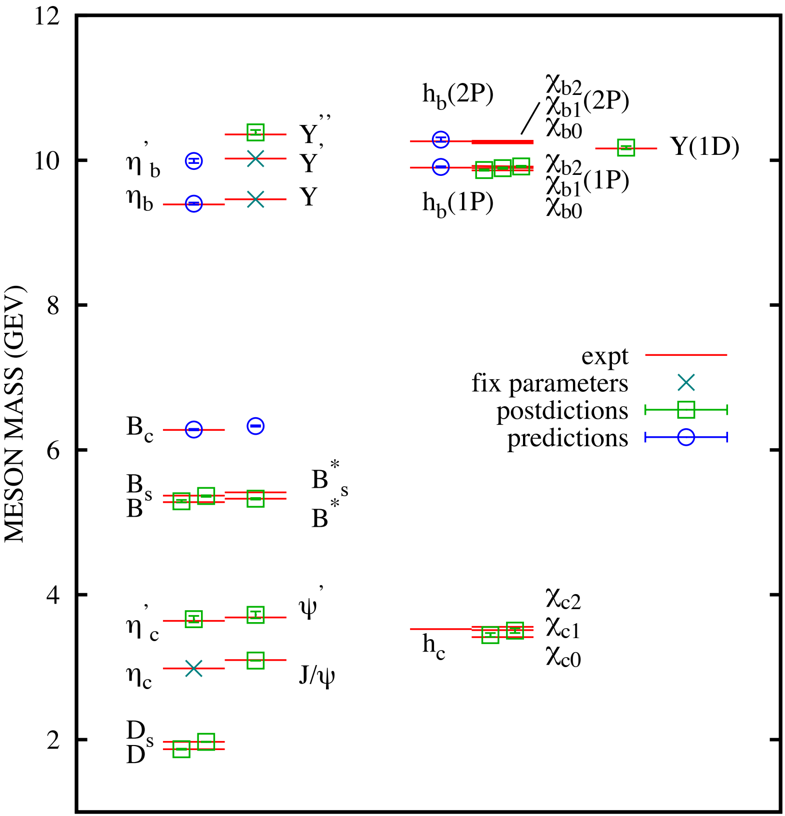
[Leinweber et al. 2003, click here for homepage]



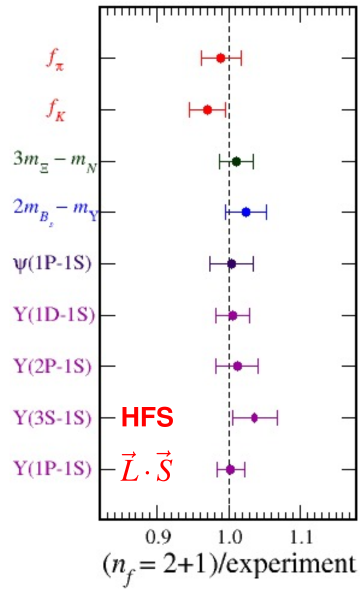
Action Density: “Pure-Glue” Vacuum Fluctuations [Leinweber et al. 2003, click here for homepage]



QCD Precision Spectroscopy: Quarkonia & Heavy-Light Mesons



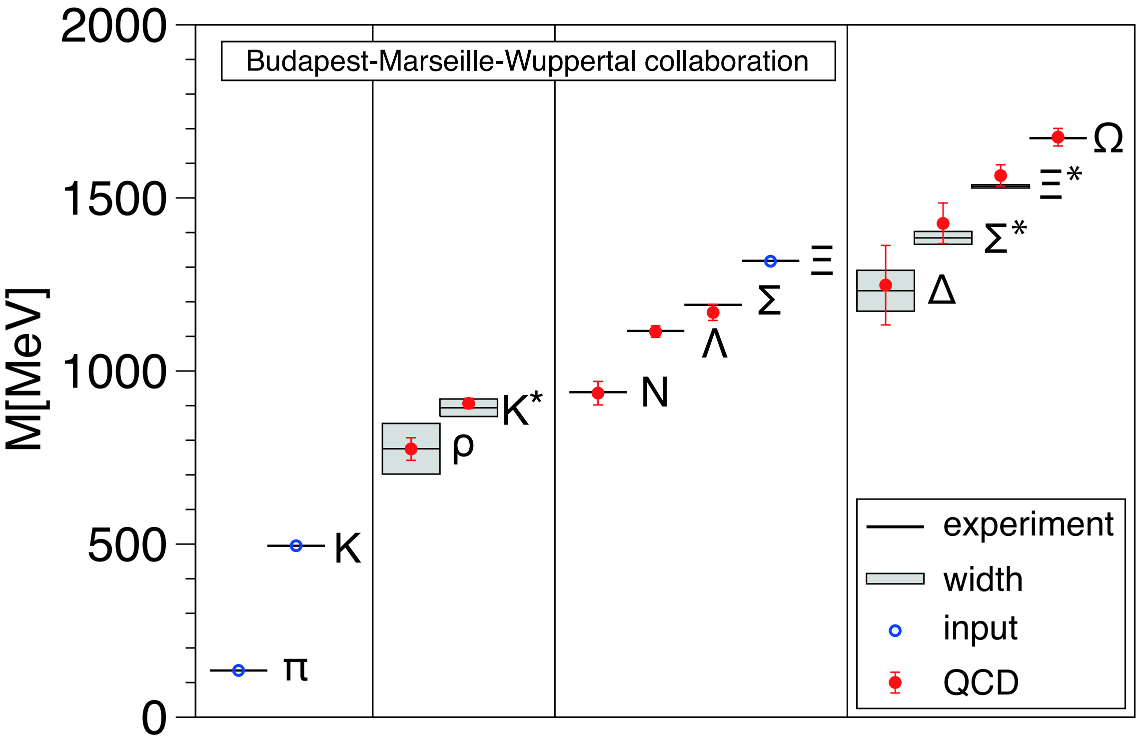
$\eta_c, \Upsilon', \Upsilon$ **set scales of** $m_c,\ m_b,\ \alpha_s(Q_0^2)$



[PDG 2013 Fig. 14.8]

QCD Spectroscopy: Systems With Light Quarks

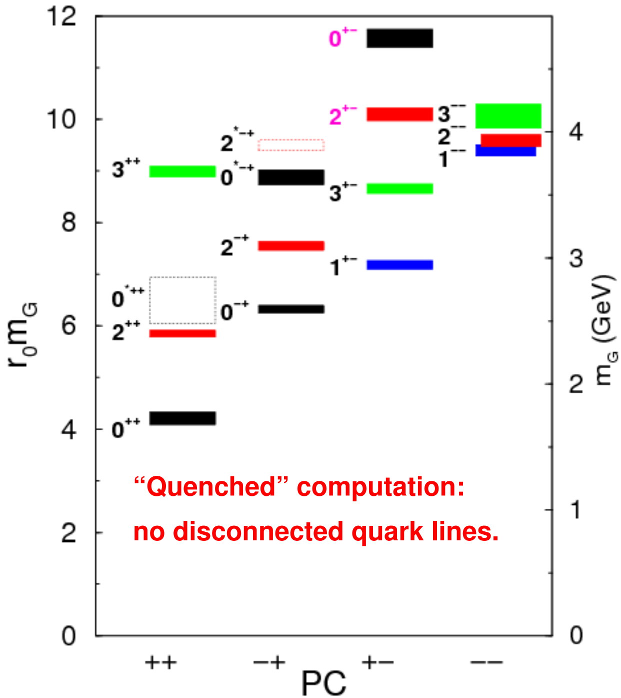
Approaching physical pion masses, good accuracy.



[Duerr et al. **Science** (2008), from PDG 2013 Fig. 14.7]

QCD Spectroscopy: Glueballs

Colour-neutral bound states of glue are *unique signal* of Non-Abelian Gauge Theories.



[Morningstar/Peardon Phys. Rev. D**60** (*1999!*) 034509]

Glueballs: Any state dominated by glue.

In particular when glue dictates **quantum numbers**.

Discovery would allow direct test of QCD

– Does not exist in Constituent Quark model!

Problem: Light-quark admixture

$\Longrightarrow$ Lattice computation:

Bad signal-to-noise,

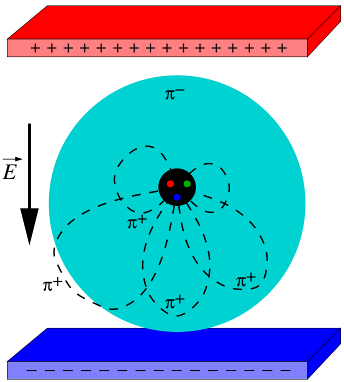
quark loops give huge corrections!

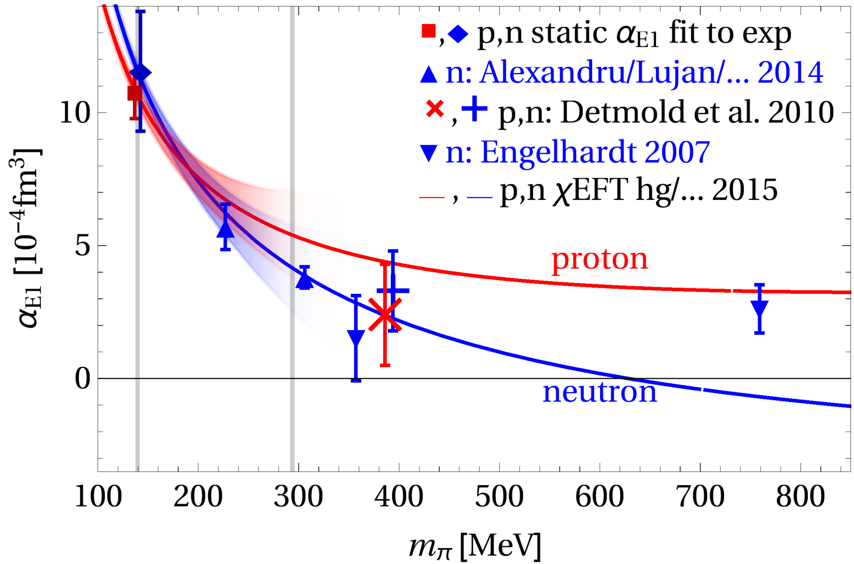
GlueX: Glueball search in Hall D is major motivation for JLab $12\mathrm{GeV}$-upgrade.

Unique experimental signal difficult.

Hadron Polarisabilities: GW Leads Connecting Data & QCD GW focus

Needs to be phrased as energy-difference: $\Delta E = -2\pi\alpha_{E1}^{(N)}\vec{E}^2$.





Neither Approach Uses The Other To Fit!

[lattice: Lujan/Alexandru/Freeman/Lee [arXiv:1411.0047 [hep-lat]];
chiral extrapolation: hgrie/McGovern/Phillips [arXiv:1511.01952 [nucl-th]];
Downie/Feldman take data at HIγS, MAMI,. . .]

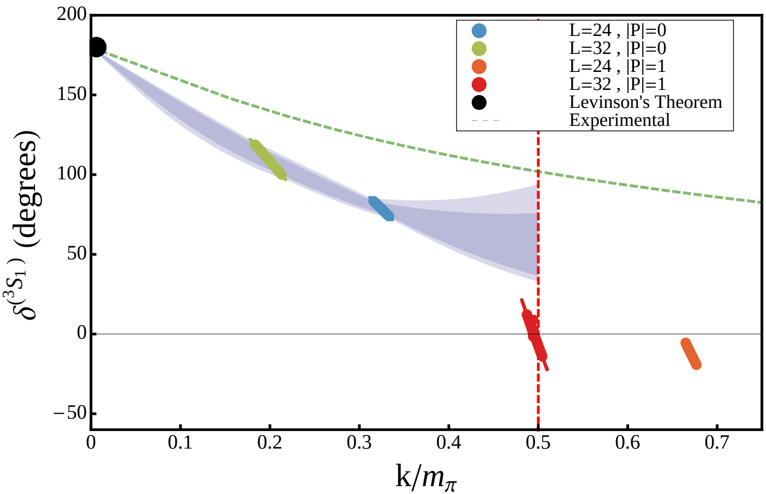
Phase Shifts Via Energy-Shift: Lüscher's Method (30 sec)

Problem: Lattice QCD gave up time-dependence by rotation to Euclidean time.

Solution: can still "feel" interactions in finite volume (cf. t-independent scattering theory)

e.g. $\pi\pi$ scattering: **compute** $\Delta E = E - 2m_\pi = 2\sqrt{\vec{k}_n^2 + m_\pi^2} - 2m_\pi \implies$ get $\vec{k}_n$, insert into

Lüscher's formula $k_n \cot\delta(k_n) = \frac{1}{\pi L} \lim_{\Lambda\to\infty} \sum_{\vec{j}}^{|\vec{j}|\leq\Lambda} \frac{1}{\vec{j}^2 - \left(\frac{k_n L}{2\pi}\right)^2} - 4\pi\Lambda$ (with error bars!)



NN scattering at $m_\pi = 805\text{MeV}$ [NPLQCD PR**C88** (2013) 124003]

Valid for: below first inelasticity,

$L \gg$ interaction range $r_0 \sim \frac{1}{m_\pi}$

but can have scatt. length $a \sim L$!

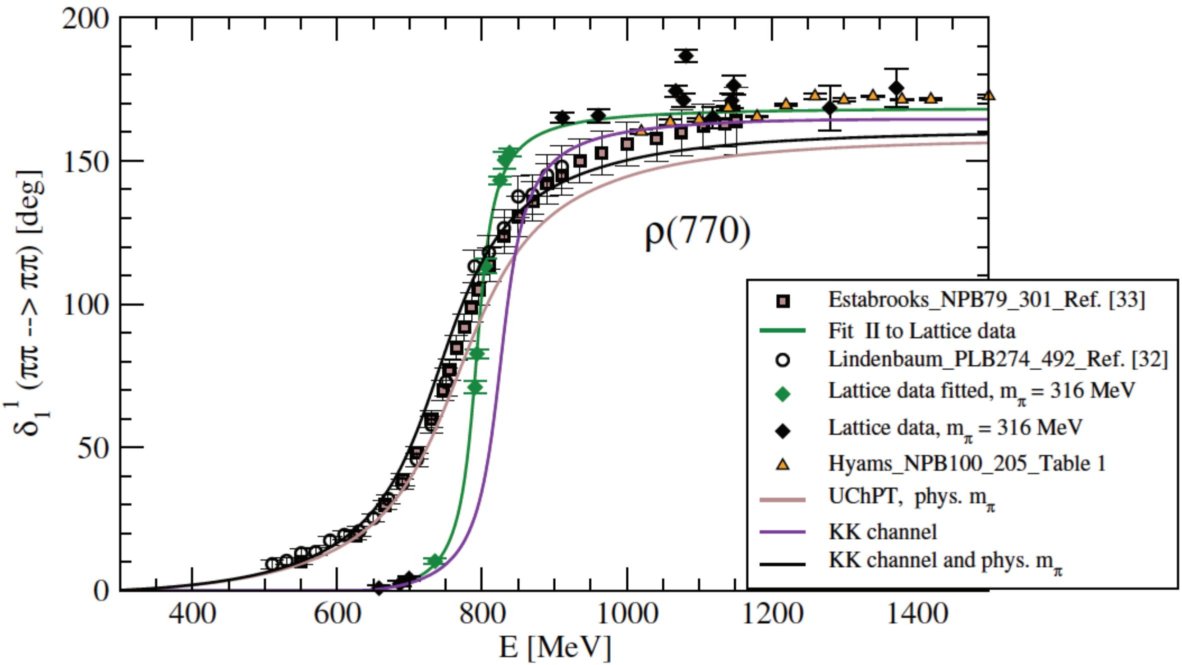
Many extensions available and being worked on:

3-body, box with different lengths, coupled channels,...

[Döring/Mai/..., hg...]

Phase Shifts Computed Via Energy-Shift: Tiny Effect

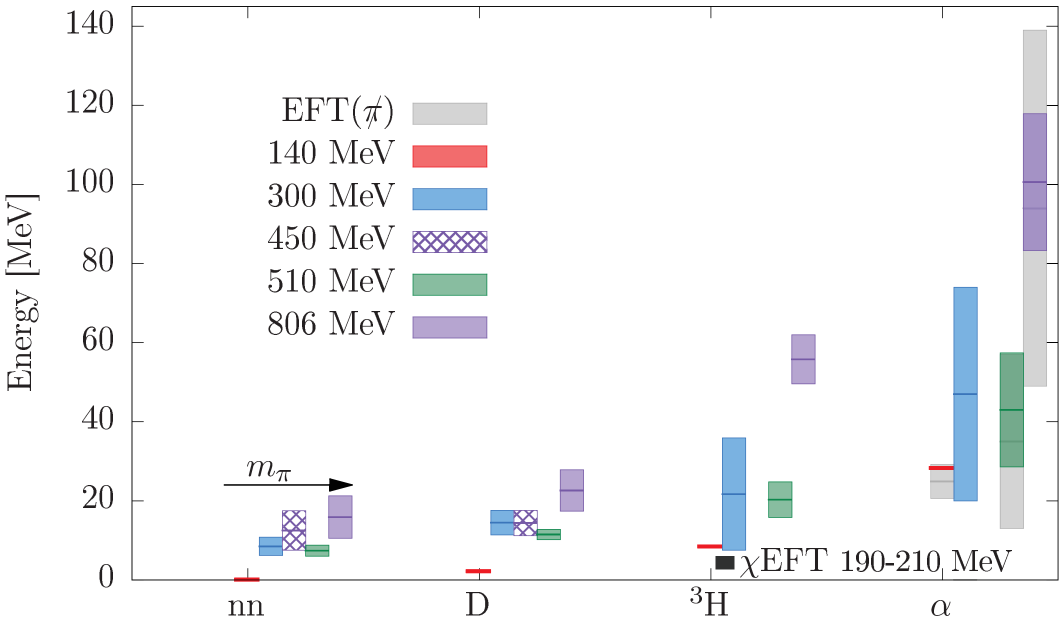
$\pi\pi$ phase shifts identify ρ resonance; unphysical $m_\pi = 316\text{MeV} > m_\pi^{\text{phys}} = 140\text{MeV}$.



Alternative Worlds: Lightest Nuclei at Higher Pion Masses

NPLQCD
HALQCD

Merger of EFT and lattice has started exploring how few-nucleon systems emerge from QCD.



[J. Kirscher [arXiv:1509.07697 [hep-lat]] (got his PhD in GW's EFT group)]

Surprisingly little change in few-nucleon systems – but nn becomes bound when m_π increased!

Next: 4. Pions and 0, 1, 2,. . . Nucleons

Familiarise yourself with: [(Goldstone: CL 5; Ryd 8.1-3);

CL 5; Ryd 8.1-2; Ber 2, 3;

Scherer/Schindler: Primer χEFT;

Ericson/Weise: Pions and Nuclei Chap. 9– see me!]