PHYS 6610: Graduate Nuclear and Particle Physics I

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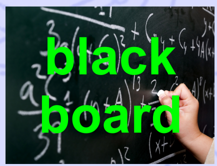
I. Tools

5. A Quantum Field Theory Reminder

Or: You Won't Learn It Here!

References: [HH QM-II; Edyn Fields; Ryder 2-4; extended notes on Canonical Quantisation from NuPa-II]



Quotes

The reason Dick's physics was so hard for ordinary people to grasp was that he did not use equations. The usual way theoretical physics was done since the time of Newton was to begin by writing down some equations and then to work hard calculating solutions of the equations... He had a physical picture of the way things happen, and the picture gave him the solution directly with a minimum of calculation. It was no wonder that people who had spent their lives solving equations were baffled by him. Their minds were analytical; his was pictorial.

F. Dyson on R. Feynman [Kaku: QFT]

I was sort of half-dreaming, like a kid would... that it would be funny if these funny pictures turned out to be useful, because the *damned Physical Review would be* [emphasising the words, laughing incredulously] *full* of these odd-looking things. And that turned out to be true.

R. Feynman [Kaku: QFT]

(g) Spin-$\frac{1}{2}$ Fermions & QED: $\mathcal{L}_{\rm QED} = \overline{\Psi}\left[\mathrm{i}\left(\not\partial + \mathrm{i}q\not{A}\right) - m\right]\Psi$

Summary: Dirac Matrices and Free Dirac Spinors

$\{\gamma^\mu,\gamma^\nu\} = 2\,g^{\mu\nu}$ and $\pm 2m\,P_\pm = \not{p} \pm m = \sum\limits_{s=\pm} \begin{matrix} u_s(p)\bar{u}_s(p) \\ v_s(p)\bar{v}_s(p) \end{matrix}$ of solutions $\begin{matrix} [\not{p}-m]u_s(p)=0 \\ [\not{p}+m]v_s(p)=0 \end{matrix}$.

of free (on-shell) field

$$\Psi(t;\vec{r}) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \sum_{s=\pm} \left[b_s(\vec{p})\, u_s(p)\, \mathrm{e}^{-\mathrm{i}px} + d_s^\dagger(\vec{p})\, v_s(p)\, \mathrm{e}^{\mathrm{i}px}\right] \delta^{(4)}(p^0 - \sqrt{\vec{p}^2 + m^2}):$$

$\begin{matrix} b_s(\vec{p}) \\ d_s(\vec{p}) \end{matrix}$ annihilates *one* $\begin{matrix} \text{particle} \\ \text{anti-particle} \end{matrix}$ with $\vec{p}$, spin s; $\begin{matrix} b_s^\dagger(\vec{p}) \\ d_s^\dagger(\vec{p}) \end{matrix}$ creates *one* $\begin{matrix} \text{particle} \\ \text{anti-particle} \end{matrix}$ with $\vec{p}$, spin s.

Dirac Spinors Ψ, u_s, v_s: objects/states in 4-dimensional, abstract **Spinor space** (basis e.g. combined spin and (anti-)particle states). $\alpha = 1,2,3,4$th component $\Psi^\alpha, (u_s)^\alpha, (v_s)^\alpha$.

Dirac Matrices γ^μ**:** vector in **Minkowski Space** of SRT, with each a 4×4 matrix in **Spinor space**.
Components $(\gamma^\mu)^\alpha{}_\beta$ are components $\mu = 0,1,2,3$ of vector in **Minkowski Space** of SRT,
with each a component of a 4×4 matrix with components $(\alpha\beta)$ in **Spinor space**, $\alpha,\beta = 1,2,3,4$.

$$\Longrightarrow \{\gamma^\mu,\gamma^\nu\} = 2\,\mathbb{1}_{\rm spinor}\, g^{\mu\nu}_{\rm Minkowski} \quad \text{or} \quad \left(\{\gamma^\mu,\gamma^\nu\}\right)^\alpha{}_\beta = 2\,\delta^\alpha_\beta{}_{\rm spinor}\, g^{\mu\nu}_{\rm Minkowski}$$

Never *need* explicit representation of Dirac matrices & spinors, but some convenient for *understanding*.

Dirac/Standard Representation of Dirac Matrices: γ^0 Diagonal

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;,\; \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \implies \gamma^5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\Psi := \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}$: $\quad \mathcal{L}_{\text{QED}} = \left(\Psi_+^\dagger, \Psi_-^\dagger\right) \begin{pmatrix} E + qA_0 - m & -\vec{\sigma}\cdot(\vec{p} - q\vec{A}) \\ -\vec{\sigma}\cdot(\vec{p} - q\vec{A}) & E + qA_0 + m \end{pmatrix} \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}$

$\Psi_\pm$ “like” non-relativistic (Pauli’s 2-dim./bi-) $\begin{matrix}\text{particle} \\ \text{anti-particle}\end{matrix}$ spinors with poles at $\begin{matrix} E = m > 0 \\ E = -m < 0 \end{matrix}$ for $\vec{p} = 0$.

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$\Psi_\pm$ "like" non-relativistic (Pauli's 2-dim./bi-) particle / anti-particle spinors with poles at $E = m > 0$ / $E = -m < 0$ for $\vec{p} = 0$.

Chiral Basis of Dirac Matrices: $\vec{\gamma}$ Diagonal

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \;,\; \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \implies \gamma^5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $\Psi := \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix}$: $\mathcal{L}_{QED} = \left(\varphi_R^\dagger, \varphi_L^\dagger\right) \begin{pmatrix} E + qA_0 + \vec{\sigma}\cdot(\vec{p} - q\vec{A}) & \mathbf{m} \\ \mathbf{m} & E + qA_0 - \vec{\sigma}\cdot(\vec{p} + q\vec{A}) \end{pmatrix} \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix}$

Chiral Basis of Dirac Matrices: $\vec{\gamma}$ Diagonal

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \,,\; \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \implies \gamma^5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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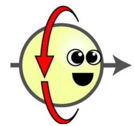
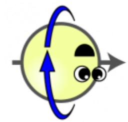
Bi-spinors φ_{RL} are eigenstates to **helicity operator** $h := \dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$



vs.



$\begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix}$ eigenfunctions to spin projection on $\vec{p}$, eigenvalues $\begin{matrix} +1 \\ -1 \end{matrix}$ when spin $\begin{matrix} \text{parallel} \\ \text{anti-parallel} \end{matrix}$ to $\vec{p}$.

cf. left-/right- circularly polarisation of photons.

For $m = 0$ (i.e. $E = |\vec{p}|$), φ_R and φ_L do not mix. $\implies$ conserved quantum number.

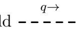
for $m > 0$, can "overtake" fermion in flight $\implies \vec{p}$ reversed $\implies$ helicity reversed! $\implies$ not conserved!

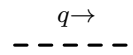
$\implies$ helicity mixing $\sim \dfrac{m}{E} \sim 1 - \beta$ $\nearrow$ 1 for non-relativistic ($E \approx m$) $\implies$ not conserved at all; $\searrow$ 0 for ultra-relativistic ($E \gg m$) $\implies$ approximately conserved

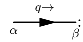
Interactions with gauge fields A^μ (QED,...) *conserve helicity* since do not mix φ_{LR}! (see later)

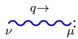
(h) Handout: Popular Feynman Rules (Link to website here.)

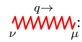
Propagators

nonrelativistic (boson or fermion) field  : $\mathrm{i}\Delta_\mathrm{F}^\mathrm{nonrel}(q) = \dfrac{\mathrm{i}}{q_0 - \frac{\vec{q}^2}{2m} + \mathrm{i}\epsilon}$

real/complex scalar  : $\mathrm{i}\Delta_\mathrm{F}(q) = \dfrac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\epsilon}$

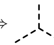
fermion Dirac $\alpha\beta$  : $\mathrm{i}S_\mathrm{F}(q) = \dfrac{\mathrm{i}\,[\not{q} + m]^\beta{}_\alpha}{q^2 - m^2 + \mathrm{i}\epsilon}$

photon ($\partial \cdot A = 0$: Lorenz gauge)  : $\mathrm{i}\Delta_\mathrm{F}^{\mu\nu}(q) = \dfrac{-\mathrm{i}\,g^{\mu\nu}}{q^2 + \mathrm{i}\epsilon}$

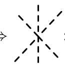
massive spin-1  : $\mathrm{i}\Delta_\mathrm{F}^{\mu\nu}(q) = \dfrac{-\mathrm{i}\,(g^{\mu\nu} - \frac{q^\mu q^\nu}{m^2})}{q^2 + \mathrm{i}\epsilon}$

Interactions and Their Lagrangeans

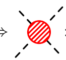
Φ^n Theories: Interactions of Scalar With Itself

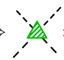
$-\dfrac{\lambda_3}{3!}\Phi^3 \Longrightarrow$  : $-\mathrm{i}\,\dfrac{\lambda_3}{3!}$

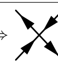
real: $-\dfrac{\lambda_4}{4!}\Phi^4$; complex: $-\dfrac{\lambda_4}{2!^2 = 4}(\Phi^\dagger\Phi)^2$ $\Longrightarrow$  : $-\mathrm{i}\,\dfrac{\lambda_4}{4! \text{ or } 4}$

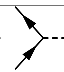
real: $-\dfrac{\lambda_6}{6!}\Phi^6$; complex: $-\dfrac{\lambda_6}{3!^2}(\Phi^\dagger\Phi)^3$ $\Longrightarrow$  : $-\mathrm{i}\,\dfrac{\lambda_6}{6! \text{ or } 3!^2}$

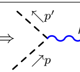
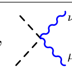
Some self-interactions with derivatives (all p incoming):

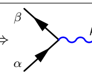
via s-wave: $-4\lambda_\mathrm{s}\left[\Phi^\dagger\Phi\right]\left[\Phi^\dagger(\partial^2\Phi) - (\partial^2\Phi^\dagger)\Phi\right] \Longrightarrow$  : $-\mathrm{i}\,\lambda_\mathrm{s}\sum\limits_{i=1}^{4} p_i^2$

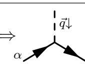
via p-wave: $-6\lambda_\mathrm{p}\left[(\partial_\mu\Phi)^\dagger(\partial^\mu\Phi)\right]^2 \Longrightarrow$  : $-\mathrm{i}\,\lambda_\mathrm{p}\sum\limits_{i=1}^{3}\sum\limits_{j=i+1}^{4} p_i \cdot p_j$

4-Fermion/Fermi-like Theory: $-\lambda_\mathrm{F}\,(\overline{\Psi}\Psi)^2 \Longrightarrow$  : $-\mathrm{i}\,\lambda_\mathrm{F}$

Yukawa(-like) Theory: fermion-scalar $f_0\overline{\Psi}\Phi\Psi + f_5\overline{\Psi}\gamma_5\Phi\Psi$ (+ H.c. if Φ complex) $\Longrightarrow$  : $\mathrm{i}f_0 + \mathrm{i}f_5\,\gamma_5$

Scalar QED: $-\mathrm{i}eA^\mu\left[(\partial_\mu\Phi^\dagger)\Phi - \Phi^\dagger(\partial_\mu\Phi)\right] + e^2A^\mu A_\mu\Phi^\dagger\Phi \Longrightarrow$  : $-\mathrm{i}\,e(p^\mu + p'^\mu)$,  : $\mathrm{i}\,e^2 g^{\mu\nu}$

(Fermionic) QED: $-e\overline{\Psi}\not{A}\Psi \Longrightarrow$  : $-\mathrm{i}\,e\,(\gamma^\mu)^\beta{}_\alpha$

πN Toy-Model Pauli bi-spinor $N = \binom{\uparrow}{\downarrow}$ with spin $\vec{\sigma}$: $-\dfrac{g_A}{2f_\pi}N^\dagger\left(\vec{\sigma}\cdot(\vec{\partial}\pi)\right)N \Longrightarrow$  : $\dfrac{g_A}{2f_\pi}\,(\vec{\sigma})^\beta{}_\alpha \cdot \vec{q}$

(i) A More Realistic Set of Interactions

Low-Energy Compton Scattering off Nucleons

$$\mathcal{L}_{\pi}^{(2)} = \frac{1}{2}\,\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + eA_\mu\,\epsilon_{3ij}\,\pi_i\partial^\mu\pi_j + \frac{1}{2}\,e^2A_\mu A^\mu(\pi_1^2+\pi_2^2) - \frac{1}{2}m_\pi^2\vec{\pi}^2 + \dots$$

$$\mathcal{L}_{\pi N}^{(1)} = N^\dagger(\mathrm{i}v\cdot D + g_A u\cdot S)N$$

$$\mathcal{L}_{\pi N}^{(2)} = N^\dagger\left[\frac{1}{2M_\mathrm{N}}\Big((v\cdot D)^2 - D^2 - \mathrm{i}g\{S\cdot D, v\cdot u\}\Big) + 4c_1 m_\pi^2\left(1-\frac{1}{2f_\pi^2}\pi^2\right)\right.$$
$$\left. + \left(c_2 - \frac{g_A^2}{8M_\mathrm{N}}\right)(v\cdot u)^2 + c_3 u\cdot u - \frac{\mathrm{i}}{4M_\mathrm{N}}[S^\mu,S^\nu]eF_{\mu\nu}\big((1+\kappa^{(\mathrm{s})}) + (1+\kappa^{(\mathrm{v})})\tau_3\big)\right]N + \dots$$

$$\mathcal{L}_{\pi N}^{(4)} = 2\pi e^2 N^\dagger\left[\frac{1}{2}\left(\delta\beta^{(\mathrm{s})} + \delta\beta^{(\mathrm{v})}\tau_3\right)g_{\mu\nu} - \left((\delta\alpha^{(\mathrm{s})}+\delta\beta^{(\mathrm{s})}) + (\delta\alpha^{(\mathrm{v})}+\delta\beta^{(\mathrm{v})})\tau_3\right)v_\mu v_\nu\right]F^{\mu\rho}F^{\nu}{}_{\rho}$$

$$\mathcal{L}_{\gamma\mathrm{N}\Delta}^{\mathrm{HB},(2)} = -\frac{\mathrm{i}eb_1}{M_\mathrm{N}}\left(H^\dagger S_\rho F^{\mu\rho}\Delta_\mu^3 - (\Delta_\mu^3)^\dagger S_\rho F^{\mu\rho}H\right)$$

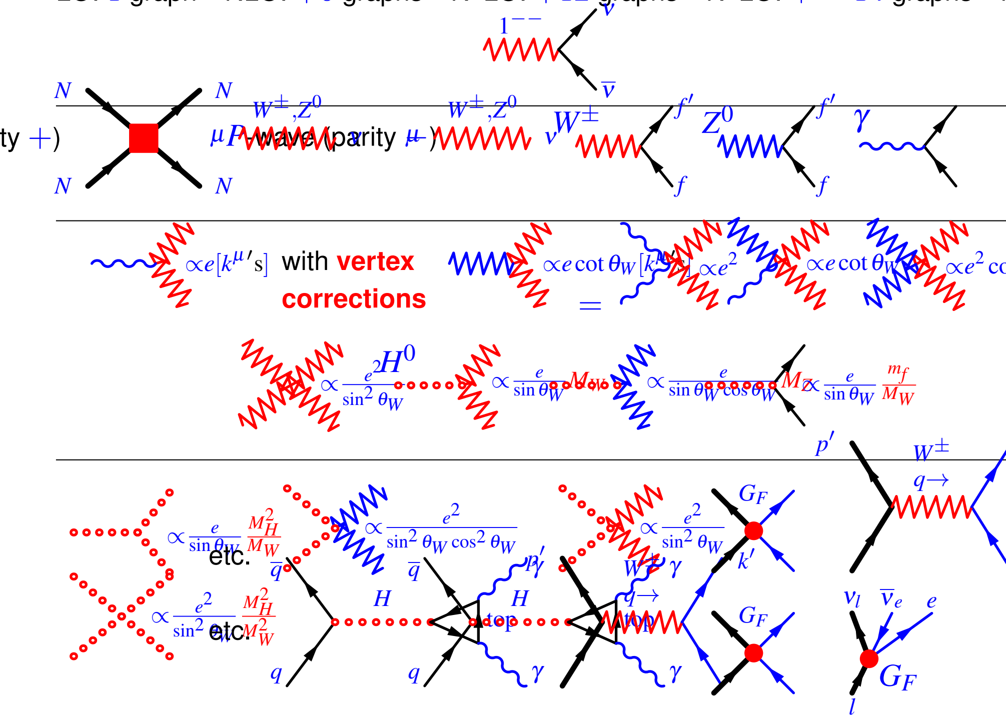
$$\mathcal{L}_{\gamma\mathrm{N}\Delta}^{\mathrm{HB},(3)} = -\frac{eb_2}{2M_\mathrm{N}^2}H^\dagger\left(S\cdot\overleftarrow{D}\,F^{\mu\rho}v_\rho + F^{\mu\rho}v_\rho S\cdot D\right)\Delta_\mu^3 + \frac{eD_1}{4M_\mathrm{N}^2}\left[H^\dagger v^\alpha S^\beta\langle\tau^3[D_\mu,f_{\alpha\beta}^+]\rangle\Delta_\mu^3 + \mathrm{H.c.}\right]$$

$$\mathcal{L}_{\gamma\mathrm{N}\Delta}^{\mathrm{PP},(2)} = \frac{3e}{2M_\mathrm{N}(M_\mathrm{N}+M_\Delta)}\left[\bar{N}(\mathrm{i}g_M\tilde{F}^{\mu\nu} - g_E\gamma_5F^{\mu\nu})\partial_\mu\Delta_\nu^3 - \bar{\Delta}_\nu^3\overleftarrow{\partial}_\mu(\mathrm{i}g_M\tilde{F}^{\mu\nu}g_E\gamma_5F^{\mu\nu})N\right]$$

All 1N Contributions to N^{4}LO for $\omega \lesssim 300$ MeV

[Bernard/Kaiser/Meißner 1992-4, Butler/Savage/Springer 1992-3, Hemmert/... 1998
McGovern 2001, hg/Hemmert/Hildebrandt/Pasquini 2003, McGovern/Phillips/hg 2013]

LO: 1 graph NLO: $+0$ graphs N^{2}LO: $+12$ graphs N^{3}LO: $+\sim 14$ graphs N^{4}LO: $+\sim 60$ graphs



$e^2\delta^0$ **LO**

$e^2\delta^2$ N^{2}LO

$e^2\delta^3$ N^{3}LO

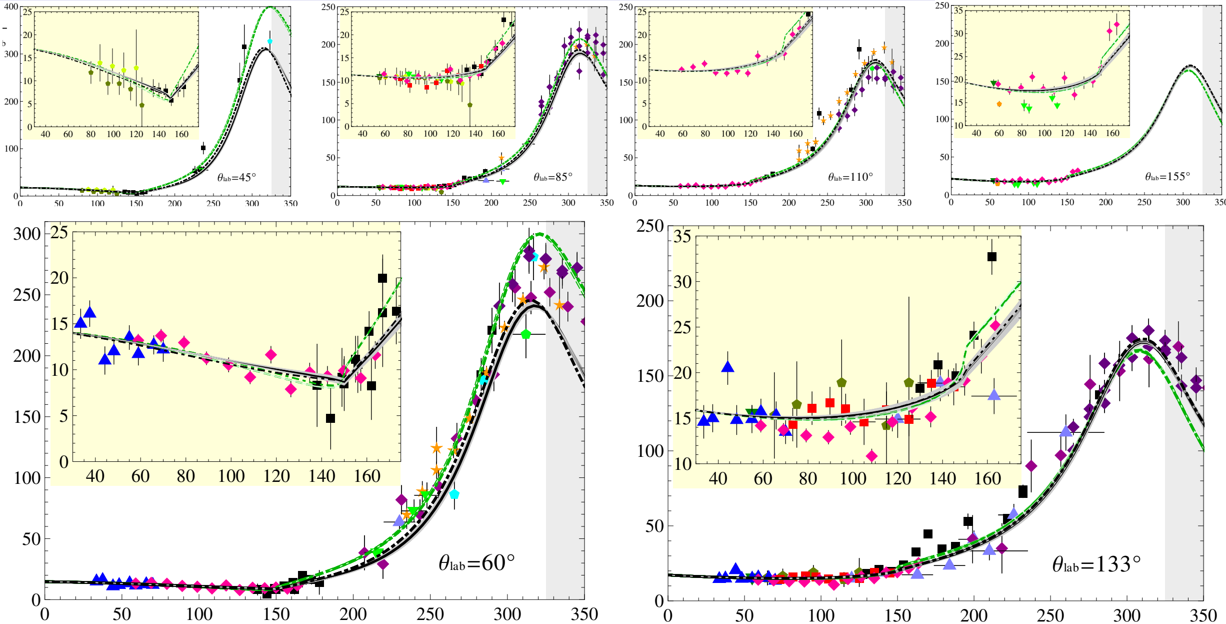
$e^2\delta^3$ N^{3}LO

$e^2\delta^4$ N^{4}LO

Diagrams obtained by permuting or crossing are not displayed.

Comparison to Experiment

data: MAMI, HIγS,… (Briscoe/Downie/Feldman/…)



Proton polarisabilities extracted from data:

α_{E1}^{p} $[10^{-4}\ \text{fm}^3]$	β_{M1}^{p} $[10^{-4}\ \text{fm}^3]$	χ^2/d.o.f.
$10.65 \pm 0.4_{\text{stat}} \pm 0.2_{\Sigma} \pm 0.3_{\text{theory}}$	$3.15 \mp 0.4_{\text{stat}} \pm 0.2_{\Sigma} \mp 0.3_{\text{theory}}$	$113.2/135$

Next: 6. Discrete Symmetries: T, C, P

& 7. Scattering and Decay

Familiarise yourself with: [TCP: HG 9.1-5, MM, HH – Scatt+Decay: HH; HG 10.1-2, 5.7/12; PRSZR 4; HM 4.3, 2.10, 4.4; PDG 47, 47.5, 48]