Problem Sheet 9

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.

Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at home.gwu.edu/~hgrie/lectures/nupa-I15/nupa-II15.html.

1. One-Loop Boson Self-Energy in the Yukawa(-Like) Theory (continued) (4P, plus 2P still attributed to HW 8): At the end of the last HW, we calculated the one-loop renormalised boson propagator in dimensional regularisation, with the renormalised masses $M_R$ for the fermion, $m_R$ for the boson, and $\epsilon = (4 - d)/2$: This corrects wrong factors in my handwritten solution!

$$-i\Sigma_{\Phi(\Psi)}(p) = - \require{cancel} \ImaginaryPart[p]$$

$$\Sigma_{\Phi(\Psi)}(p) = \frac{g_R^2}{4\pi^2} \left[ \left( M_R^2 - \frac{p^2}{6} \right) \left( \frac{1}{\epsilon} - \gamma_E + 2 \right) - \int_0^1 dx \left[ M_R^2 - p^2 x (1 - x) \right] \ln \frac{M_R^2 - p^2 x (1 - x) - i\epsilon}{4\pi\mu^2} \right]$$

a) (2P still attributed to HW 8) Adjust the wave function and mass normalisation of the scalar such that (only) the “infinity” $\frac{1}{\epsilon}$ is absorbed in them, rendering the total self-energy (1-loop plus counter term) finite. This is the Minimal Subtraction Renormalisation Condition.

Hint: Do not do any integrals – it is possible to answer the following questions qualitatively!

b) (1P) Show: This propagator has no imaginary part for $p^2 \leq 4M_R^2$, except for the customary $i\epsilon$.

c) (3P) Show: For a range of $x$ inside $[0; 1]$, the denominator of this propagator develops a positive imaginary part for $p^2 > 4M_R^2 \geq 0$. When the scalar is on-shell, $p^2 = m_R^2$, then it can decay into a fermion-antifermion pair.

Note beyond this problem: Therefor, the pole wanders out into the complex $p$ plane with increasing $p^2 > 0$, as indicated by the blue arrows in the figure. It’s important that the imaginary part has the same sign as the customary $i\epsilon$. Only then does causality hold.

2. Renormalisability in 4 Dimensions (6P): Let’s write down the most generic interactions between $n$ bosons (scalar, vector, photon, whatever) and $m$ fermions (quarks, electrons, whatever), with $l$ derivatives. There are of course very many structures with $\gamma$ matrices and the like, but symbolically, we can write $\mathcal{L} = \lambda_{(lmn)} \Psi^m \partial^{l} \Phi^n$.

a) (3P) Identify all triplets $(lmn)$ for which the interaction is super-renormalisable and give examples of actual interactions we discussed in class – or make some up.

b) (3P) Identify all triplets $(lmn)$ for which the interaction is renormalisable and give examples.

Please turn over.
3. **Infinite Part of the One-Loop Vertex Correction in the Yukawa(-like) Theory (9P):**

This is both a good exercise and a nice illustration how short a problem can get when one does a bit of thinking. We consider again the Yukawa-like theory with real scalar \( \Phi \) (mass \( m \)) and Dirac spinor \( \Psi \) (mass \( M \)), coupled by \(-g\bar{\Psi}\Phi \Psi\), and \( \epsilon = (4 - d)/2 \).

a) **(3P)** Find all superficially divergent diagrams at one-loop.

b) **(5P)** One of these is the vertex correction \( p \rightarrow \rightarrow p' : -ig_{\text{R}} \mu^\epsilon \Gamma[p, p'] \).

Calculate its divergent part only, i.e. only the part which goes like \( \frac{1}{\epsilon} \). Yes, this is a three-propagator integral, but it gets easier because you only want the divergent part. First, throw away all terms which do not have enough powers of \( q \) in the numerator to lead to a UV divergence. Then, do the remaining Dirac algebra. Finally, you can get rid of one of the propagators by partial fractions \( \frac{A}{A-B} = 1 + \frac{B}{A-B} \) and using again that you are only after divergences.

c) **(1P)** Determine the coupling renormalisation \( Z_g \) by BPHZ renormalisation in the MS scheme.

4. **Different Renormalised Propagators of the Yukawa(-like) Theory (11P):** Yes, \(-g\bar{\Psi}\Phi \Psi\) again! Now, you do not do another loop integral but dissect its result for massless bosons, \( m = 0 \).

With \( \frac{1}{\epsilon} := \left( -\frac{2}{4 - d} - \gamma_E + \ln 4\pi \right) \), the one-loop self-energy correction and its BRST Counter Terms are:

\[
\sum\phi(p) = -\frac{g_{\text{R}}^2}{2(4\pi)^2} \left[ \left( \frac{3M_R}{2} + \frac{\phi - M_R}{2} \right) \frac{1}{\epsilon} - \int_0^1 dx \left( M_R(1 + x) + (\phi - M_R)x \right) \ln \left( \frac{1 - x(M_R^2 - \phi^2)}{\mu^2} \right) \right]
\]

and the renormalised propagator:

\[
\frac{i}{\phi - M_R + \left( (Z_{\Phi} - 1)(\phi - M_R) - A - \sum\phi(p) \right)}
\]

For your convenience, I have already started to split the result as series in powers of \( (\phi - M_R) \).

You may wish to follow closely the QED treatment. To speed up writing, you can drop sub- and superscripts, but keep in mind the point of the exercise: Renormalised quantities are scheme-dependent.

**Do not perform any Feynman parameter integrals – leave them be!**

a) **(2P)** Determine the wave function renormalisation \( Z_\Psi^{\text{MS}} \) and mass shift \( A^{\text{MS}} = M_B - M_R^{\text{MS}} \) in the Modified Minimal Subtraction prescription which absorbs only \( \frac{1}{\epsilon} \) (including the infinity \( \frac{1}{\epsilon} \)).

b) **(4P)** Find \( Z_\Psi^{\text{OS}} \) and \( A^{\text{OS}} \) in the On-Shell Scheme, i.e. when the renormalised propagator has a pole at \( M_R^{\text{OS}} \) with residue i. Recall that \( p^2 = M^2 \) when \( \phi = M \) and expand in powers of \( (\phi - M) \):

\[
\Sigma(p) = \Sigma(M_R^{\text{OS}}) + (\phi - M_R^{\text{OS}})\Sigma + \text{rest}.
\]

c) **(3P)** Show: The propagator contains only “physical” parameters in OS, but depends on \( \mu \) in MS.

d) **(2P)** The renormalised mass differs in the two schemes. Find the transformation between them:

\[
M_R^{\text{OS}} = M_R^{\text{MS}} + \delta M(\text{OS} \rightarrow \text{MS})
\]

**Comment beyond the scope:** This shows that the pole of the MS propagator lies not at \( \phi = M_R^{\text{MS}} \), but at \( \phi = M_R^{\text{MS}} + \delta M(\text{OS} \rightarrow \text{MS}) = M_R^{\text{OS}} \). So the pole of the propagator is identical at this order in both schemes – as Physics requires: A pole can be seen in an amplitude!

You also see that the wave function renormalisations are different. The propagator in MS does not have residue i – you need to heal that by multiplying amplitudes with powers of \( \delta Z(\text{OS} \rightarrow \text{MS}) \).