Problem Sheet 7

Due date: 5 March 2015 13:00

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.

Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.

News and .pdf-files of Problems also at home.gwu.edu/~hgrie/lectures/nupa-I15/nupa-II15.html.

1. **Dirac Matrices (5P):** Prove these results for the traces over Dirac matrices, using the Dirac algebra \( \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu} \), symmetry properties of the trace, \( \gamma_5^2 = 1 \) and \( \{ \gamma_5, \gamma^\mu \} = 0 \) (see [HH QM, chap. 10]):

   a) (1P) \( \text{tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu} \)

   b) (2P) \( \text{tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4 [g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}] \)

   c) (2P) The trace over an odd number of \( \gamma \) matrices is zero.

   **Hint:** As usual, you are allowed to consult a good book.

2. **Massive Spin-1 Field (9P):** The Lagrangean of a massive real spin-1 field \( B^\mu \) looks like that for the photon, but with a mass term. Another term describes coupling to an external current \( j^\mu \):

   \[
   L_{\text{spin-1}} = -\frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{m^2}{2} B_\mu B_\mu - B_\mu j^\mu
   \]

   a) (3P) Determine the equations of motion (called Proca-equations). Which condition on \( B^\mu \) guarantees that the current \( j^\mu \) is conserved? How many independent polarisations does \( B^\mu \) have?

   b) (3P) Show: The propagator is \( iD_{F}^{\mu\nu}(k) = \frac{-i}{k^2 - m^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2} \right] \) and obeys current conservation on-shell.

   c) (3P) One would expect that one retrieves the photon propagator for \( m \to 0 \) — but one runs into trouble. Show: The reason is that the part of \( L \) which is quadratic in \( B^\mu \) has a zero mode for \( m = 0 \), for any \( k^\mu \). Remember a) and that matrices with a zero eigenvalue can not be inverted.

3. **Symmetry Factors and Feynman Rules (8P):**

   a) (3P) As hinted, symmetry factors are the origin of many a discussion. In “\( \Phi^4 \) Theory”, the interaction between 4 real and complex scalar fields is written with different numerical factors,

   \[
   L_{\text{real}} \Phi = -\frac{\lambda}{4!} \Phi^4 \quad \text{while} \quad L_{\text{complex}} \Phi = -\frac{\lambda}{4} (\Phi^\dagger \Phi)^2,
   \]

   but most books cite both Feynman rules as \( \bigotimes \): -i\( \lambda \). Why?

   b) (3P) Derive the Feynman rule of the interaction between a Pauli-spinor \( \psi \) (2-dimensional vector in spin space) and a real scalar field \( \pi \) via the three Pauli matrices \( \sigma^a \):

   \[
   L_{\pi N} = -\frac{g_A}{2f_\pi} \psi^\dagger \left( \vec{\sigma} \cdot \vec{\partial} \pi \right) \psi
   \]

   This is a simplified version of the lowest-order interaction between a non-relativistic nucleon (i.e. no anti-nucleon) and the pion field. One finds \( g_A = 1.267 \) and \( f_\pi = 92.21 \text{ MeV} \) [PDG 2012].

   c) (2P) In the natural system of units, the action has mass dimension zero. Show that the dimensions of the coupling constants in b) guarantee that \( L_{\pi N} \) has the right mass dimension.

Please turn over.
4. A Particular Form of Cutoff Regularisation (8P): In anticipation of the discussion of regularisation procedures, consider the one-loop integral

\[ \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m^2 + i\varepsilon} \rightarrow \int_{-\infty}^{\infty} \frac{dq_0}{(2\pi)} \int_{q^2 < \Lambda^2} \frac{d^3q}{(2\pi)^3} \frac{1}{q^2_0 - q^2 - m^2 + i\varepsilon}. \]

This regularisation breaks Lorentz invariance, as momenta are cut off by $\Lambda$ while energies are not.

a) (4P) Calculate the integral. The “i$\varepsilon$” is crucial to tell you where the poles in the energy integration are relative to the real axis.

b) (4P) Expand the result for $m \ll \Lambda$ and compare to the corresponding result in dimensional regularisation. Which terms look alike, which are absent? Are the terms which are absent analytic in $m^2$? And why could that be important?