Problem Sheet 12

Due date: 19 April 2018 **12:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at home.gwu.edu/~hgrie/lectures/nupa-18I/nupa-18I.html.

1. Pion Decay and Parity (**5P**): Another classic weak process (see e.g. [PRSZR]; value [PDG 2016]):

$$\frac{\Gamma[\pi^+ \to e^+\nu_e]}{\Gamma[\pi^+ \to \mu^+\nu_\mu]} = [1.230 \pm 0.004] \times 10^{-4}$$

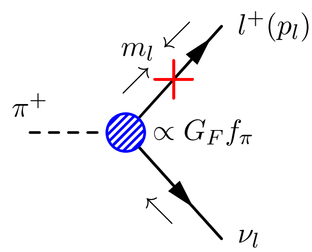
According to the Scattering chapter, the pion decay rate at rest into two leptons $l\nu_l$, each carrying a momentum of magnitude p_l, is related to the matrix element $\mathcal{M}$ and phase-space factor by:

$$\Gamma[\pi^\pm \to l + \nu_l] = \frac{p_l}{8\pi m_\pi^2}\, |\mathcal{M}_l|^2 \quad .$$

a) (**1P**) Show: If one would assume that $\mathcal{M}_l$ is the same for the decay into a muon or electron, then the phase space would actually prefer the latter by a ratio of about 2 to 5.

b) (**1P**) Show: If leptons were massless, then helicity conservation would forbid these decays. A few words accompanying a figure with the spins and helicities of the particles suffices.

Context: We now discuss evidence for the vector-nature of the weak interaction ("V-A theory"). Helicity is flipped by a mass term. In the diagram, "+" is the mass insertion, i.e. we treat the particle mass as a small perturbation with its own Feynman rule ($m_l \ll p_l$). The arrows next to the lines give the particle helicities. The blue- shaded dot "hides" the W^+ production and annihilation in the low-energy version of the GSW theory (coupling $\propto G_F f_\pi$ because of pion decay and V-A).



It is not too difficult to show that (Do *not* show, except if you want 5**P extra credit**!)

$$|\overline{\mathcal{M}}|^2 = A\, G_F^2\, f_\pi^2\, m_l^2\, p_l\, m_\pi \quad .$$

c) (**3P**) Show: A has mass-dimension zero. Show: This formula explains the ratio of decay rates found experimentally (see PDG).

2. Decay of the $W^\pm$ Boson (**3P**): In HW 8.3, you found the decay width of a massive vector particle into massless fermions: $\Gamma = \dfrac{b^2 M}{12\pi}$, with coupling $-ib\,\gamma^\mu$. We will now translate this to the decay of the $W^\pm$ boson. Do not do the calculation explicitly. Rather, use this result you already have.

First, relate b to the coupling constant between the $W^\pm$ and a lepton pair – it was given in the lecture; see slide " (Tree Level) Interactions and Experimental Numbers". Keep in mind the number of helicity-pairs in the decay of HW 8.3 and in the GSW coupling. Finally, insert numbers and compare to the experimental value, $\Gamma[W^+ \to e^+\nu_e] \approx 220$MeV. ***Error estimate!***

Please turn over.

3. DO IT YOURSELF (**12P**): Nothing creates understanding better than going over material yourself. Plus, who knows if I have all the signs in the script right – so I put the problem to the "bee-hive mind". In that spirit, let's explore the details of the Higgs-Kibble-Englert mechanism of dynamical gauge boson mass generation in the Landau-Ginzburg model. The Lagrangean involves the complex scalar field Φ and the $U(1)$ gauge boson A_μ, with real parameters $\lambda > 0$, $a > 0$:

$$\mathcal{L} = [(\partial_\mu + \mathrm{i}\, eA_\mu)\,\Phi]^\dagger\, [(\partial^\mu + \mathrm{i}\, eA^\mu)\,\Phi] - \lambda \left[\Phi^\dagger\Phi - a^2\right]^2$$

   a) (**1P**) Why can the term which is quadratic in Φ only *not* be interpreted as mass?

   b) (**1P**) The scalar acquires (for "reasons unknown") a vacuum expectation value. Show that when $\lambda > 0$, $|\Phi(x)| = a$ is indeed a *minimum* of the potential for all space-time points x.

   c) (**2P**) The real scalar (Higgs) field $\varphi(x)$ parametrises x-dependent fluctuations about the minimum:

$$\Phi(x) = \left[a + \frac{1}{\sqrt{2}}\,\varphi(x)\right] \mathrm{e}^{\mathrm{i}\pi(x)/(\sqrt{2}a)} \quad .$$

   The original Lagrangean is invariant under $U(1)$ gauge transformations

$$\Phi(x) \to \mathrm{e}^{-ie\alpha(x)}\Phi(x) \quad , \quad A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x) \quad , \quad \alpha(x) \text{ any real function.}$$

   Find how $\varphi(x)$ and $\pi(x)$ transform under gauge transformations. Show that one can always perform a gauge transformation to set $\pi(x) \equiv 0$ at all points. So, this field is irrelevant.

   d) (**3P**) Now let us compare $\mathcal{L}$ in the new parametrisation ($\pi(x) \equiv 0$) with the free massive real scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m_\phi^2}{2}\phi^2 \quad .$$

   Check that the kinetic energy terms of the real fields ϕ and φ are identical. Determine the masses of the Higgs field φ and of the gauge field A_μ in terms of e, λ, a. Be mindful that mass-squares are positive, as required. If not, you have some signs wrong.

   e) (**2P**) State the Feynman rules for all 4 interactions in the new Lagrangean.

   f) (**3P**) **And now for the difference to the Goldstone Theorem**: Take the same Lagrangean, but without gauge field ($e \equiv 0$), so there is no gauge symmetry either. Now, you *cannot* eliminate the field $\pi(x)$ in the reparametrisation of $\Phi(x)$ – because there is no local symmetry you can use. Determine the mass of the φ and π fields. Find all interactions which involve $\pi(x)$. Which of the Feynman rules contribute when the π field has zero momentum? [Just draw diagrams, do not write down the corresponding Feynman rules.]