Problem Sheet 11

Due date: 12 April 2018 **12:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at home.gwu.edu/~hgrie/lectures/nupa-18I/nupa-18I.html.

1. Lattice Derivatives (**10P**): The simplest total lattice action for a real massive scalar field is obtained from the continuum version by just replacing derivatives with differences, e.g. in 1 dimension:

$$\frac{1}{2}\int \mathrm{d}x \ \left[(\partial_\mu\Phi(x))\,(\partial^\mu\Phi(x)) + m^2\Phi^2(x)\right] \ \to \ \frac{1}{2}\sum_{\text{points } n}\left[\frac{1}{a}\,(\Phi(n+1)-\Phi(n))^2 + am^2\Phi^2(n)\right] \ ,$$

where $\Phi(n)$ denotes a real (i.e. real) scalar field at lattice point n, m its mass, and a the lattice spacing. We now construct the spectrum of the free theory for bosons on an infinite, one-dimensional lattice.

a) (**4P**) Take the Fourier transform $\Phi(n) = \int \frac{\mathrm{d}k}{(2\pi)}\ \mathrm{e}^{ikna}\Phi(k)$ of the lattice action and show that the lattice action in momentum space is

$$\frac{1}{2}\int \frac{\mathrm{d}k}{(2\pi)}\ \Phi(-k)\left[\frac{4}{a^2}\ \sin^2\frac{ak}{2} + m^2\right]\Phi(k) \ .$$

Since this relation has period $2\pi/a$ in k, momenta lie in the *first Brillouin zone*, $-\frac{\pi}{a} \leq k \leq \frac{\pi}{a}$.

b) (**2P**) In the QFT Section, we saw that the piece of the action which is quadratic in the fields is the inverse propagator. So you have just derived the inverse propagator on the lattice ("energy-momentum dispersion relation"). Compare it in one plot to its continuum limit.

c) (**4P**) Now consider a fermion field on a 1-dimensional, infinite lattice (with Dirac matrix γ_0):

$$\sum_{\text{points } n}\bar{\Psi}(n)\left[\gamma_0\frac{\Psi(n+1)-\Psi(n-1)}{2} + am\ \Psi(n)\right] \ ,$$

Repeat the steps above, compare the resulting lattice dispersion relation to the continuum limit, and show that momenta k and $\frac{\pi}{a}-k$ contribute equally to the dispersion relation $E^2 = f(k)^2+m^2$.

2. Neutrino Helicity (**6P**): The Goldhaber-experiment (1958) is key to the GSW theory, and I only mentioned it. Read the account given at the end of [PRSZR 18.6] and/or in [Per 7.6] (or any other source you like) and answer the following questions:

a) (**2P**) The reasoning only works if the total orbital angular momentum of the initial electron-nucleus system is zero. Why is the relative orbital angular momentum between e^- and Eu zero?

b) (**2P**) How did Goldhaber determine the direction in which the neutrino is emitted?

c) (**2P**) What rôle does the iron around the Eu source play?

Please turn over.

3. Quarkonium Decays (**4P**): The decay widths of quarkonium states into two photons or two charged leptons (via a virtual photon) are at leading order:

$$\Gamma[{}^1\mathrm{S}_0 \to 2\gamma] = 3 \times \frac{4\pi}{m_q^2}\,(Z_q^2\alpha)^2|\Psi(0)|^2 \;\;,\;\; \Gamma[{}^3\mathrm{S}_1 \to e^+e^-] = 3 \times \frac{4\pi}{m_q^2}\,(Z_q\alpha)^2|\Psi(0)|^2 \;\;,$$

and the decay widths into gluon states are at leading order

$$\Gamma[{}^1\mathrm{S}_0 \to 2g] = \frac{2}{3}\,\frac{4\pi}{m_q^2}\,\alpha_s^2|\Psi(0)|^2 \;\;,\;\; \Gamma[{}^3\mathrm{S}_1 \to 3g] = \frac{160(\pi^2-9)}{81m_q^2}\,\alpha_s^3|\Psi(0)|^2 \;\;,$$

The difference stems from the colour factors. These can be derived, but that is lengthy (do not do it). The quark mass is m_q, its charge is Z_q, and "3×" accounts for the three quark colours per flavour.

a) (**1P**) The 2016 PDG lists the $\Upsilon(9460)$ decay probability into e^+e^-, $\mu^+\mu^-$ and $\tau^+\tau^-$ pairs as 2.5%, each. Why do you expect that they are practically the same?

b) (**3P**) Since the $\Upsilon(9460)$ decays to 81.7% into three gluons, find α_s at the bottomonium scale. Provide an estimate of theoretical uncertainties, based on the fact that the above formulae capture only the leading contribution in α_s. Compare to the "running-α_s" plot of the lecture.

[translated from German: www.pp.rhul.ac.uk/~ptd/TEACHING/PH2510/pauli-letter.html

4th of December 1930

Dear Radioactive Ladies and Gentlemen,

As the bearer of these lines, to whom I graciously ask you to listen, will explain to you in more detail, how because of the "wrong" statistics of the N and Li6 nuclei and the continuous beta spectrum, I have hit upon a desperate remedy to save the "exchange theorem" of statistics and the law of conservation of energy. Namely, the possibility that there could exist in the nuclei electrically neutral particles, that I wish to call neutrons, which have spin 1/2 and obey the exclusion principle and which further differ from light quanta in that they do not travel with the velocity of light. The mass of the neutrons should be of the same order of magnitude as the electron mass and in any event not larger than 0.01 proton masses. The continuous beta spectrum would then become understandable by the assumption that in beta decay a neutron is emitted in addition to the electron such that the sum of the energies of the neutron and the electron is constant. [...]

I agree that my remedy could seem incredible because one should have seen these neutrons much earlier if they really exist. But only the one who dare can win and the difficult situation, due to the continuous structure of the beta spectrum, is lighted by a remark of my honoured predecessor, Mr Debye, who told me recently in Bruxelles: "Oh, It's well better not to think about this at all, like new taxes". From now on, every solution to the issue must be discussed. Thus, dear radioactive people, look and judge. [...]

Your humble servant,

W. Pauli