Start-Quiz   Hand in by Monday, 2 July 2018 *at the latest – the earlier the better*

Why? This quiz allows you and us lecturers to judge your math skills, and some Physics concepts. The quiz covers topics which are essential for the graduate courses but which we assume you have mostly well-understood and mastered in your undergraduate studies – although this is by far not a complete list. It will allow us to gauge how solid your background is, in order to plan the courses accordingly.

Why Math? Advanced Math skills are an essential foundation to quantitatively describe Nature. As you see from the problems, we assume that you have experience of vectors, matrices, multi-dimensional differentiation and integration, ordinary differential equations, complex numbers, etc., comparable to:
R. Shankar: Basic Training in Mathematics – A Fitness Program for Science Students; Springer 2008; except Sects. 6 (complex analysis) and 10.5-7 (partial diff. eq.'s). An excellent self-study guide.
M. L. Boas: Mathematical Methods in the Physical Sciences; 3rd ed., John Wiley 2005; roughly chaps. 2-8.
F.W. Byron and R.W. Fuller: Mathematics of Classical and Quantum Physics; reprint by Dover Publications 1992; roughly chapters 1 and 3.
The Mathematical Methods course will cover advanced techniques at a rapid pace, without time to repeat these. I will send out a list of recommended books for *PHYS6110: Mathematical Methods* later in August.

Why not more Physics? The test does not attempt to gauge your familiarity with basic concepts of Classical Mechanics, Electrodynamics and Quantum Mechanics to the same extent as your Math skills, since the Graduate Physics courses emphasise the *deductive (top-down/theory-to-experiment)* approach rather than the inductive (bottom-up/phenomena-to-theory) approach of your undergraduate courses. Phenomena will be *derived* rather than observed.

Why problems? If lecturers are asked what they covered in a course on Mathematics, the default answer is "everything, and the Path Integral". If students are asked whether they "know/have seen" matrix multiplication, the default answers are "yes" and "I have never heard of it", often in the same sentence. The latter is nearly certainly incorrect; the former does not show the degree of working knowledge. A simple problem gauges proficiency: To which degree can formal knowledge be applied?

This quiz does not count towards your grade. However, you should immediately get accustomed to hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers.

What to do? **Except for numerics problems, solve the problems with pen/pencil and paper and do *not* use electronic devices – that would defy the purpose. You may use books to refresh your memory, but make sure you understand each term and know *how* to do the problems.**
We hope you easily solve it in less than 6 hours total, but we advise you only answer a few questions at a time, without pressure. Although I do not recommend it, you can skip a problem if you are absolutely sure you can do it. Write "trivial" in your solution. If you have problems or identify areas unfamiliar to you, *describe the problem in your solution.* Examples: "Do not know how to add two numbers; what is $\vec{\nabla}$?; cannot get a convergent result; don't know Fortran."
If some of the notation or words are unclear, do not hesitate to contact me at hgrie@wu.edu.
If you spot major deficits, contact me as soon as possible and prepare for remedial self-study. The earlier you hand in the quiz, the earlier you get feed-back and can start on potential remedial study. Don't procrastinate.
To make the quiz more interesting, we included some problems which we think you *can not* do yet. But we do not tell you which. . .
So do not assume that not knowing some answers is automatically bad. Wait for our assessment. Your honest feedback is very important. This is *not* an exam. If you cheat, you only cheat on yourself.

Format: I strongly prefer scanned handwriting as .pdf file sent to hgrie@gwu.edu. Must be legible. Typed/TEX'ed *not* needed! The answers suffice, but a few lines of derivation would be much appreciated.

Participation is *strongly recommended*: With your honest assessment of your skills, you will know where you stand, and we will know where to **start lectures on Monday, 27 August 2018**.

Please turn over.

1. TEXTBOOKS USED: Which undergraduate textbooks did you use in

   (i) Mathematics/Mathematical Methods;

   (ii) Electrodynamics;

   (iii) Quantum Mechanics?

2. BASIC LOGICAL REASONING: If the Moon is made of Cheese, all Mice are Astronauts. Given this sentence, decide which of the following statements is a logical consequence in the abstract world of Mice, Moons, Cheese and Astronauts – no connection to the real world.

   a) When the Moon is not made of Cheese, Mice cannot be Astronauts.

   b) Although one Mouse is not an Astronaut, the Moon is made of cheese.

3. MATRICES: Given the matrix $\mathcal{M} = \begin{pmatrix} \frac{\lambda+3}{4} & \frac{\lambda-1}{4} & \frac{\lambda-1}{2\sqrt{2}} \\ \frac{\lambda-1}{4} & \frac{\lambda+3}{4} & \frac{\lambda-1}{2\sqrt{2}} \\ \frac{\lambda-1}{2\sqrt{2}} & \frac{\lambda-1}{2\sqrt{2}} & \frac{\lambda+1}{2} \end{pmatrix}$, with $\lambda \in \mathbb{C}$ arbitrary.

   a) Determine all eigenvalues and normalised eigenvectors. **Hint:** At least one eigenvalue is 1.

   b) Determine a (the?) matrix which diagonalises $\mathcal{M}$ and write down the diagonalised version of $\mathcal{M}$.

   c) Determine the inverse of $\mathcal{M}$. Are there λ for which the inverse does not exist?

   d) **If you cannot do problems a) to c):** Do it for the 2×2 matrix made of the upper left of $\mathcal{M}$.

   e) More generally: Is the trace of a matrix invariant under unitary transformations?
      Hint: This problem is independent of a) to c).

4. ROTATE the vector $\vec{r} = (1, 2, 3)$ counter-clockwise about the y-axis by $60°$.

5. PAULI-MATRICES: Every invertible 2×2 matrix can be written in the form $A = \alpha \mathbf{1} + \sum_{i=1}^{3} \beta_i \sigma^i$, where α, β_i are real numbers, $\mathbf{1}$ is the 2×2 unit matrix and σ^i are the Pauli spin matrices. True or false?

6. BASIC ANALYSIS:

   a) Calculate the first two non-trivial terms (i.e. the first two terms which are not just pure numbers) in the Taylor series around $x = 0$ of: (i) $\dfrac{1}{\sqrt{1+x^2}}$, (ii) $\dfrac{\tan x}{x}$, (iii) $\dfrac{1 - \cos x}{x^2}$.

   b) Bring these expressions into the form $a + ib$, with $i^2 = -1$ the imaginary unit and a, b real:

   $$\text{(i) } \sqrt{i} \ , \quad \text{(ii) } \frac{1+i}{1-i} \ , \quad \text{(iii) } \sin \frac{\pi i}{2}$$

   c) What is the Fourier Series of $f(x) = x$ on the interval $-\pi \le x \le \pi$?

   d) Determine $\sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

   Please see next page.

7. BASIC VECTOR ANALYSIS:

   a) Show $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$.

   b) Write the vector (x, y, z) in spherical coordinates.

   c) Calculate with $\vec{r} = x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z$, $r = \sqrt{x^2 + y^2 + z^2}$, $\vec{e}_r = \frac{\vec{r}}{r}$:

   $$\text{(i) } \vec{\nabla} \cdot \vec{r}, \quad \text{(ii) } \vec{\nabla} \times \vec{r}, \quad \text{(iii) } \vec{\nabla}\frac{1}{r}$$

   d) Write down the Laplace operator $\Delta = \vec{\nabla}^2$ in spherical coordinates. Calculate $\Delta\frac{1}{r}$ everywhere.

   e) Does the field $\vec{A}(\vec{r}) = \dfrac{1}{1 + x^2}\,\vec{r}$ have sources/sinks or curls? If so, where?

   f) Write the integral $\displaystyle\int_{-\infty}^{\infty} \mathrm{d}^3 r\; f(\vec{r})$ in cylindrical coordinates. Carefully state the integration limits!

   g) A function has the form $f(x, y, z)$, where $y(x), z(x)$ implicitly depend on x. Determine the
   (i) total derivative; (ii) partial derivative of f with respect to x.

   h) Use Gauß' Theorem to show: The integral of the field $\vec{A}(\vec{r}) = \vec{r}$ over the surface of an arbitrarily-shaped volume V is given by $3V$.

   "Gauß" is the actual spelling of the German mathematician who is spelled "Gauss" in English.

   i) Verify Stokes' theorem by explicit calculation in the example field $\vec{A} = (x^2 y, 2yz, 3xz)$. Pick as surface of integration a disk of radius R, centred at the origin in the xy-plane.

8. NOTATIONS:

   a) Describe the Einstein Summation Convention, or give an example.

   b) Given two abstract wave functions $| \Psi \rangle$ and $| \Phi \rangle$ in Dirac's bra-ket notation. Write their overlap $\langle \Psi \mid \Phi \rangle$ using their coordinate space versions $\Psi(x) = \langle x \mid \Psi \rangle$, $\Phi(x) = \langle x \mid \Phi \rangle$.

9. DIFFERENTIAL EQUATIONS:

   a) Consider the ordinary differential equation $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) + \lambda^2 x^2 = a^2$, with a, λ and $f(x)$ real. How many linearly independent solutions exist?

   b) Solve the above equation for the boundary conditions $f(x = 0) = 1$, $\frac{\mathrm{d}}{\mathrm{d}x} f(x = 0) = 0$.

   c) Find the most general solution to the ordinary differential equation $\dfrac{\mathrm{d}}{\mathrm{d}x} f(x) = x\, f(x)$.

10. BEYOND SIMPLEST MATH:

   a) Determine using the residue theorem: $\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\; \dfrac{1}{x^2 + a^2}$, $a > 0$.

   b) Solve $\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\; (2x + 3)\, \delta(2x - 3)$, where $\delta(x)$ is Dirac's delta function.

   c) Sketch how you would solve the following partial differential equation: $\left(\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + \dfrac{\mathrm{d}^2}{\mathrm{d}y^2} \right) f(x, y) = 0$.

   d) Does the set of even numbers form a group under additions?

Please turn over.

11. BASIC CLASSICAL MECHANICS:

    a) Write down the Lagrange function for the harmonic oscillator and derive the equation of motion.
    b) Explain the Principle of Least Action in 1 or 2 sentences.

12. BASIC ELECTRODYNAMICS:

    a) Determine the electric potential and electric field in the exterior *and* interior of a sphere with radius R and total charge Q for a spherically symmetric charge distribution $\rho(\vec{r}) = \mu\, r^n$, $n > -3$.
    b) Which experiment would you perform, standing outside the sphere, to determine the charge distribution inside the sphere? You are not allowed to touch the sphere.
    c) How many linearly independent polarisations does a free, monochromatic light-wave have?

13. BASIC QUANTUM MECHANICS: Let's look at the infinite-square well of unit width, i.e. $V(x) = 0$ for $x \in [0; 1]$, infinite otherwise.

    a) Write down the normalised ground-state wave-function.
    b) Is the operator $p_x x$ Hermitean? Is $[p_x, x]$ Hermitean?
    c) What are the ground-state expectation values of the operators in b)?
    d) Which of the following wave functions is a solution to the problem? (A is a suitable normalisation constant.) Why/not?

$$\text{(i) } A\left[\sin \pi x + \sqrt{2}\sin 4\pi x\right] \quad, \quad \text{(ii) } A\exp-(x-1)^2$$

14. PROGRAMMING SKILLS: If you do not know *either* of the two languages C or Fortran, or no LaTeX, or no symbolic manipulation programme, remedial self-study is very strongly recommended.

    a) Did you have a numerical methods course, and which textbook did you use?
    b) Rank your level of proficiency for the following programming languages:
       (i) Fortran: expert, good, novice, none
       (ii) C: expert, good, novice, none
       (iii) an algebraïc manipulation and plotting programme like Maple, Mathematica, Matlab, Reduce: expert, good, novice, none – which?
       (iv) the scientific text-processing programme LaTeX: expert, good, novice, none
    c) Which other programming languages did you learn?
    d) NUMERICAL APPROXIMATION OF A SECOND DERIVATIVE: Write a program (in Fortran or C) which numerically evaluates the second derivative of $f(x) = e^{-x^2}\cos(x)$, where $x = \frac{\pi}{7}$. Do so by differencing the first forward and backward derivatives of $f(x)$.
    e) SUM A SERIES: Consider the doublet of positive integers (n, m) from which we form the series

$$S(m,n) = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{i^2 + j^2} \ .$$

    Write a program to compute this sum and answer the following questions:
       i. What conditions do m and n have to satisfy to get convergence at the 1 part-per-million level? How many iterations are required for this accuracy?
       ii. Plot $S(m,n)$ as a function of m and n from $S(1,1)$ to $S(100,100)$.

15. ANOTHER VERY IMPORTANT QUESTION: Which line of research are you interested in?