Problem Sheet 12

Due date: 28 November 2018 **16:00**

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Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

1. Heating a Ring: Continuation (**7P**): In the previous HW, we constructed the Green's function to the heat equation on a ring:

$$D\,T(x) = Q(x) \quad \text{and} \quad T(x=0) = T(x=1),\; T'(x=0) = T'(x=1) \;\;,\; x \in [0;1]$$

by using the spectral decomposition of the differential operator $D = -\mathrm{d}_x^2$. Now, we test another way to construct the Green's function. You (should) have shown before that the Green's function is not quite the inverse of D, but that

$$D_x\,G(x;y) = \delta(x-y) - 1 \;\;.$$

a) (**2P**) Show that both $G(x;y)$ and $T(x)$ have no zero mode, just like $Q(x)$, cf. a) in the last HW.

b) (**1P**) As you know, $\frac{1}{2}|x-y|$ is a Green's function to $+\mathrm{d}_x^2$, but to the wrong boundary conditions. We therefore write an ansatz

$$G(x;y) = -\frac{1}{2}|x-y| + f_h(x) \;\;.$$

Which explicit condition(s) does $D_x\,f_h(x)$ have to fulfill?

c) (**4P**) Construct now explicitly $G(x;y)$ from the ansatz. Take full advantage of reciprocity, boundary conditions and constraints on $G(x;y)$.

2. Frobenius' Method for Hermite Polynomials (**6P**): Hermite polynomials appear in the quantum-mechanical harmonic oscillator wave-function as solutions to the Hermite equation

$$f''(x) - 2x\,f'(x) + 2\nu\,f(x) = 0 \;\;.$$

We can solve this with a simplified Frobenius ansatz $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$, i.e. $\alpha = 0$.

a) (**1P**) Show that the indicial equations dictate that $\alpha = 0$ or $\alpha = 1$.

b) (**2P**) Derive the recurrence relation for the coefficients a_j. Show that the solutions consists of two separate elementary solutions, comprised of either even or odd functions in x.

c) (**3P**) As you may know from your undergraduate QM class, the wave function of the time-independent harmonic oscillator is $\psi(x) \propto H_n(x)\mathrm{e}^{-x^2/2}$ and must be normalisable, i.e.

$$\langle H_n|H_m\rangle_{w(x)=\exp -x^2} := \int\limits_{-\infty}^{\infty} \mathrm{d}x\; H_n(x)\,H_m(x)\,\mathrm{e}^{-x^2} < \infty \;\;.$$

Show that this imposes that ν must have certain values, and construct the finite Hermite polynomial for $\nu = 2$, up to a normalisation constant.

Please turn over.

3. SPHERICAL BESSEL FUNCTIONS (**10P**) are the bread-and-butter of Scattering Theory in Quantum Mechanics II and arise in Mathematics as solutions to the spherical Bessel equation
$$\left[\frac{1}{x}\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,x+1-\frac{l(l+1)}{x^2}\right]f(x)=0\ ,$$
with $l\in\mathbb{N}_0^+$ a non-negative integer which will be the angular momentum in the last sub-problem.

   a) (**2P**) Find the recurrence relation for the coefficients a_j of a Frobenius ansatz around $x=0$. You should find $\nu=l$ or $\nu=-(l+1)$, and that solutions are either even or odd in x.

   b) (**1P**) Show that the series *cannot* terminate ever for non-negative integer l. This is a counter-example to the impression you may have gotten that Frobenius series must terminate.

   c) (**1P**) Depending on your choice of ν in a), you find two classes of solutions: With some normalisation choice which we will not deal with, those regular at the origin are the REGULAR SPHERICAL BESSEL FUNCTIONS $j_l(x)$ (also called sph. Bessel of the first kind); those which diverge at the origin are the IRREGULAR SPHERICAL BESSEL FUNCTIONS $y_l(x)$ (or von-Neumann functions, sph. Bessel of the second kind, also denoted as $n_l(x)$). Determine (up to a constant) the leading non-trivial behaviour of the regular and irregular solution as $x\to 0$.

   d) (**2P**) Construct closed-form expressions for $j_0(x)$ and $y_0(x)$ (up to an overall normalisation).

   e) (**4P**) An electron with energy E and wave-function $\Psi_{nlm}(r,\theta,\phi)=R_{nl}(r)\;Y_{lm}(\theta,\phi)$ is trapped in a spherical cavity of radius a with impenetrable walls, i.e. $V(r,\theta,\phi)=0$ for $r<a$ and ∞ for $r>a$. Using what you learned above, calculate all energy-levels of wave-functions which are *spherically symmetric* only. You do not need to normalise Ψ, but you may have to re-scale r.

4. MULTIPOLE MOMENTS ON THE SURFACE OF A SPHERE (**7P**): The following potential is given on the surface of a sphere with radius R:
$$\Phi(\vec{R})=\frac{\Phi_0}{2R^2}\,\left(3(\vec{\mathrm{e}}_z\cdot\vec{R})^2-\vec{R}^2\right)\ ,$$
where $\vec{\mathrm{e}}_z$ is the unit vector in the z-direction, $\vec{R}$ the radial vector from the centre of the sphere to a point to the surface and Φ_0 some constant. The origin is at the centre of the sphere.

   a) (**5P**) Determine the potential *both inside and outside* the sphere, and the first three spherical multipole moments (monopole, all dipoles and all quadrupoles) with respect to the given coordinate system. Recall that the potential should disappear at infinity and be finite as $r\to 0$.

      Hint: Upon close inspection, the potential has a form similar to one of the Spherical Harmonics.

   b) (**2P**) Determine the charge distribution on the surface of the sphere which generates the given potential. Sketch! Careful: There is an electric field both inside and outside the sphere.