Problem Sheet 11

Due date: 14 November 2018 **16:00**

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Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

1. STURM-LIOUVILLE THEORY (**10P**): For this very important problem, you can profit greatly by being inspired by textbooks. Consider the one-dimensional second-order differential operator

$$L(x) := p_2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + p_0(x) \;\;,$$

where $p_i(x)$ are three given real functions of x which (we assume) have no singularities inside the open domain $x \in]a;b[$, with a, b two real numbers which also can be $\pm\infty$. Singularities *can* occur at a and/or b. We equip the Hilbert space with the "standard" scalar product: $\langle f|g\rangle = \int\limits_a^b \mathrm{d}x\; f^*(x)\,g(x)$.

a) (**3P**) Show: $\frac{\mathrm{d}}{\mathrm{d}x}p_2(x) = p_1(x)$ is a *necessary* condition for the operator to be Hermitean. Recall that you need to check $\langle g|Lf\rangle = \langle Lg|f\rangle +$ surface terms. We will deal with the surface terms below, so keep them in mind.

b) (**1P**) Show: If L is Hermitean, then $L = \frac{\mathrm{d}}{\mathrm{d}x}p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)$. Determine p, q from the p_i's.

c) (**3P**) Such Hermitean operators L often arise in variational problems. Find a functional whose minimisation leads to an Euler-Lagrange equation of the form $L\,f(x) = g(x)$.

d) (**3P**) You showed in a) that in order for L to be Hermitean,

$$p(x)\,g^*(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) = p(x)\,f(x)\frac{\mathrm{d}}{\mathrm{d}x}g^*(x) \;\text{ at } x = a, b \;\;.$$

Discuss now under which conditions on f, g and their derivatives the operator L is self-adjoint when: (i) $p(a) = p(b) = 0$; (ii) $p(a) \neq 0$, $p(b) \neq 0$; (iii) the special case $f(a) = f(b) = 0 \wedge f'(a) = f'(b) = 0$.

2. YUKAWA POTENTIAL AS GREEN'S FUNCTION (**6P**): If the photon had a mass m, the Poisson equation of Electrostatics would read

$$\left[\Delta - m^2\right]\Phi(\vec{r}) = -4\pi\,\rho(\vec{r})$$

So, the resulting Green's function can be interpreted as the scalar potential of a "unit" point charge if the photon had a mass.

a) (**4P**) Construct a Green's function for this case, assuming that there are no charges or surfaces around. The scalar potential shall vanish at infinity. Name the boundary condition this imposes.

Hint: One may for example first construct the solution to $\left[\Delta - m^2\right] G(r,0) = 0$, and then determine the integration constants, keeping in mind the boundary conditions and that singularities or dis-continuities in G have to be avoided.

b) (**2P**) What is the solution Φ, given ρ? Is it unique?

Please turn over.

3. One-Dimensional Green's Function and the Superposition Principle (**7P**): The superposition principle is very useful even for very simple charge distributions. You know the solution to this problem inside-out, so enjoy a slightly different angle: Consider a homogeneously charged, infinitely long but infinitesimally thin, perfectly conducting plate with constant surface charge density σ, aligned in the (xy)-plane.

   a) (**1P**) Constrain the form of the field $\vec{E}$ by symmetry arguments and reduce the problem to a one-dimensional one, with x the distance from the plate.

   As we discussed in the lecture, the Green's function to the one-dimensional operator $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ "without boundaries at infinity" is

   $$G(x;x') = \frac{1}{2}\,|x-x'|\ .$$

   b) (**3P**) Calculate the electrostatic potential Φ and $\vec{E}$ for given surface charge density σ.

   Fail-safe point: For a plate located at x_0, the scalar potential is from $\Delta\Phi = -4\pi\rho$, up to a constant:

   $$\Phi(x) = \begin{cases} -2\pi\sigma(x-x_0) & \text{for } x > x_0 \\ 2\pi\sigma(x-x_0) & \text{for } x < x_0 \end{cases}$$

   c) (**3P**) Determine now the electric field between and on either side of two parallel, homogeneously charged, infinitely long, perfectly conducting plates with constant surface charge densities σ_1, σ_2.

4. Heating a Ring: Green's Functions with Zero Modes (**7P**): A time-independent heat $Q(x)$ is applied to an infinitesimally thin ring, parameterised by $x \in [0;1]$. With the differential operator $D = -\mathrm{d}_x^2$, the heat equation and boundary conditions for the temperature $T(x)$ are therefore

   $$D\,T(x) = Q(x) \quad \text{and} \quad T(x=0) = T(x=1),\ T'(x=0) = T'(x=1)\ .$$

   In this problem, we will see that a zero eigenvalue of a differential operator can constrain the solutions, because the differential operator is "not quite" invertible: If you apply it to the Green's function, you will not get a δ-distribution. In this case, the Green's function is *not* the inverse to the differential operator, but only to the eigenspaces of D to non-zero eigenvalues. Such Green's functions without zero modes are called "modified Green's functions".

   a) (**1P**) Show that the boundary conditions imply that a solution exists only if $\int\limits_0^1 \mathrm{d}x\ Q(x) = 0$, i.e. if Q has no zero mode.

   b) (**5P**) Find the spectral representation of the "inverse" of D, i.e. its eigenfunctions and eigenvalues (in coordinate space), namely the Green's function $G(x;y)$. Pay particular attention to the zero mode (i.e. to the eigenfunctions with eigenvalue zero). Check whether G is really an inverse in the space of functions which are subject to the constraint of a).

   Note: For the following, it is easier to leave your result in the form of an infinite series.

   c) (**1P**) Show whether $G(x;y)$ has a zero mode in x or y, cf. a). Are you surprised by your result?